

# Unequal Error Protection Querying Policies for the Noisy 20 Questions Problem

Hye Won Chung, Brian M. Sadler, Lizhong Zheng and Alfred O. Hero

## Abstract

In this paper, we propose an open-loop unequal error protection querying policy based on superposition coding for the noisy 20 questions problem. In this problem, a player wishes to successively refine an estimate of the value of a continuous random variable by posing binary queries and receiving noisy responses. When the queries are designed non-adaptively as a single block and the noisy responses are modeled as the output of a binary symmetric channel the 20 questions problem can be mapped to an equivalent problem of channel coding with unequal error protection (UEP). A new non-adaptive querying strategy based on UEP superposition coding is introduced whose estimation error decreases with an exponential rate of convergence that is significantly better than that of the UEP repetition code introduced by Variani *et al.*, [1]. With the proposed querying strategy, the rate of exponential decrease in the number of queries matches the rate of a closed-loop adaptive scheme where queries are sequentially designed with the benefit of feedback. Furthermore, the achievable error exponent is significantly better than that of a random block code employing equal error protection.

## Index Terms

Noisy 20 questions problem, state estimation, superposition coding, unequal error protection, error exponents.

## I. INTRODUCTION

Consider a noisy 20 questions game between a player and an oracle. The player asks binary queries to the oracle and receives a noisy version of the oracle's correct answers transmitted through a binary symmetric channel with flipping probability  $\epsilon \in (0, 1/2)$ , denoted  $\text{BSC}(\epsilon)$ . The objective of the player is to estimate the value of a continuous target variable  $X \sim \text{unif}[0, 1]$ . The player can ask questions about the first  $k$  bits in the dyadic expansion of  $X$  using  $N$  queries. The central question addressed here is: What is the optimal sequence of queries to estimate the value of  $X$  with a minimum estimation error for a specified cost function? This general setup of noisy 20 questions game and the optimal query design problem is of broad interest, arising in various areas, including active learning [2], [3], optimal sensing [4] and experimental design [5], [6], with diverse applications. For example, a

Hye Won Chung (hyechung@umich.edu) and Alfred O. Hero (hero@eecs.umich.edu) are with the EECS department at the University of Michigan; Brian M. Sadler (brian.m.sadler6.civ@mail.mil) is with the US Army Research Laboratory; Lizhong Zheng (lizhong@mit.edu) is with the EECS department at MIT. This research was supported in part by ARO grants W911NF-11-1-0391 and W911NF-15-1-0479. This research will be presented in part at 2016 IEEE International Symposium on Information Theory in Barcelona, Spain.

target localization problem in a sensor network [7] can be modeled as a noisy 20 questions game where a player (agency) aims to locate a target by receiving query responses from sensors probing the region of interest.

The problem of optimal query design for the 20 noisy questions game can be categorized into two main approaches, adaptive vs. non-adaptive designs. In each approach, the sequence of questions is designed by a controller that may either use feedback (adaptive 20 questions) or operate open-loop (non-adaptive 20 questions) to formulate the sequence of questions. For the adaptive case, the controller uses noisy answers to previous questions to determine the next question posed to the oracle. For the non-adaptive case, the controller designs the sequence of queries ahead of time without access to future answers of the oracle. In general, the use of feedback in the adaptive design provides an information advantage, allowing a better error rate of convergence, but at the cost of higher query design complexity and the need for a feedback channel from the oracle to the controller.

Previous studies on 20 questions optimal query design sought to design queries that minimize the posterior uncertainty of the target variable, where uncertainty was quantified by the Shannon entropy [8], [2], [9], [7]. In these works, the utility of observation is quantified by the expected reduction of the entropy due to the observation. This reduction is equivalent to the increase in mutual information between the target variable and the observation. For adaptive sequential querying, greedy successive entropy minimization policies [9], [10] have been extensively investigated.

When the mutual information is used to design the queries, any two queries that increase the mutual information by the same amount are considered to be equally valuable, regardless of how the queries reduce the estimation error. However, when estimation accuracy is important queries minimizing the mutual information may not be equivalent. For example, when the queries are on the coefficients in the dyadic expansion of a target variable  $X$  the queries on the most significant bits (MSBs) of  $X$  may be more valuable than those on the least significant bits (LSBs) in terms of reducing the estimation error. For estimation of  $X$ , the important question is then how to design queries that are good at reducing the estimation error.

Since in the noisy 20 questions model, the queries to the oracle result in answers that are received with errors, it is desirable to provide *unequal error protection* (UEP) for MSBs vs. LSBs in order to minimize the estimation error with a limited number of queries. This paper addresses the problem of non-adaptive query design using this concept of UEP.

To develop the UEP querying policy, we exploit the close connection between the problem of optimal query design in the noisy 20 questions problem and the problem of channel coding for the classical information transmission problem. Let  $\mathcal{M} = \{0, \dots, 2^k - 1\}$  denote the set of  $2^k$  possible states of the target variable  $X$ , determined by the first  $k$  bits in its dyadic expansion. A binary query partitions the set  $\mathcal{M}$  into two disjoint subsets, one of which contains the true state of  $X$ . For adaptive sequential querying, the partition is random, depending on the answers to the previous queries, whereas for non-adaptive querying, the partition is deterministic and determined in advance. By considering the true state of the target variable as a message transmitted from the oracle to the player and the oracle's binary answers bits to the queries as a codeword, the query design problem can be mapped to an equivalent problem of channel coding. Specifically, the query design problem reduces to the channel coding with feedback for

the adaptive case and to channel coding without feedback for the non-adaptive case.

The equivalence between the query design problem and the channel coding problem allows us to apply UEP channel coding methods to design a querying strategy. Unequal error protection coding accounts for the fact that for estimation of a target variable, the errors in the most significant bits (MSBs) are much more costly than the errors in the least significant bits (LSBs). One way to provide unequal error protection is repetition coding. In repetition coding, each bit is repeatedly transmitted multiple times, the number of repetitions varying in accordance with the desired level of unequal error protection. Such a UEP repetition coding approach to the noisy 20 questions problem was considered in [1]. It was shown that the mean squared error (MSE) of this approach decreases exponentially in  $\sqrt{N}$  where  $N$  is the number of queries. The square root of  $N$  rate is smaller than the linear in  $N$  exponential rate of decrease achievable by the bisection-based adaptive 20 questions strategy [11] that corresponds to Horstein's coding scheme for a BSC( $\epsilon$ ) with perfect feedback [12].

The main contribution of this paper is to provide a new non-adaptive querying strategy based on superposition coding [13] that can provide UEP and achieve better MSE convergence rate. The proposed superposition coding strategy provides UEP for two levels of priority, i.e., a strictly better error protection for MSBs than that for LSBs. For the MSE cost function, the different importance of MSBs vs. LSBs is captured by more highly weighted decoding error probabilities on MSBs than on LSBs when they are accounted to obtain bounds on MSE. We optimize the superposition coding strategy for these relative weights on the decoding error probabilities and show that the proposed querying strategy results in MSE that decreases exponentially in  $N$ , as contrasted to  $\sqrt{N}$ , matching the error rate of the adaptive 20 questions strategy [11]. Furthermore, we achieve a better scale factor in the MSE exponent as compared to that of a random block code employing equal error protection.

The rest of the paper is organized as follows. In Section II, we review the mathematical formulation for the noisy 20 questions problem for state estimation. We highlight the connection between the query design and the channel coding problems both for adaptive sequential querying and for non-adaptive block querying. We consider the MSE and the quantized MSE as query performance measures and quantify the value of information about the state of the target variable. In Section III, we review three well-known querying policies including the adaptive bisection policy [12], non-adaptive UEP repetition policy [1], and non-adaptive block querying based on random block coding [14]. In Sections III-A, we show that the bisection policy is the optimal myopic policy among successive entropy minimization policies in reducing the minimum MSE of the target variable (Proposition 1). In Sections III-B and III-C, two representative non-adaptive policies are presented and compared in terms of UEP property and coding gain. We introduce a new non-adaptive querying policy based on superposition coding in Section IV. We show that block querying based on superposition coding provides higher error protection for MSBs than for LSBs. We then establish that the proposed non-adaptive block querying strategy achieves better quantized MSE exponent (Theorem 1) and better MSE exponent (Corollary 1) than those of a random block code. In Section V, performance of all four policies discussed in this paper are compared by analyzing the achievable error rates of convergence for the estimation errors in the number  $N$  of queries. Finally, conclusions and future directions are discussed briefly in Section VI. After presenting each lemma, we provide a brief discussion but defer the technical details of the

proofs to the Appendices.

#### A. Notations

Capital letters will represent random variables and lower case letters will represent specific realizations of those random variables. The statistical expectation operator and the indicator operator will be denoted by  $\mathbb{E}[\cdot]$  and  $\mathbb{1}(\cdot)$ , respectively. For a continuous random variable  $X$  distributed as  $p(x)$ ,  $x \in \mathbb{R}$ , the differential entropy  $h(X)$  is defined as  $h(X) = -\int p(x) \ln p(x) dx$ . For a discrete random variable  $Y$  with distribution  $p(y)$ ,  $y \in \mathcal{Y}$ , the entropy  $H(Y)$  is defined as  $H(Y) = -\sum_{y \in \mathcal{Y}} p(y) \ln p(y)$ . The entropy of a binary random variable  $Z$  distributed as Bernoulli( $\alpha$ ),  $0 \leq \alpha \leq 1$ , is denoted  $H_B(\alpha) = -\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha)$ . The Kullback-Leibler divergence between two Bernoulli distributions Bernoulli( $\alpha$ ) and Bernoulli( $\beta$ ) is denoted  $D_B(\alpha \parallel \beta) := \alpha \log \frac{\alpha}{\beta} + (1 - \alpha) \log \frac{1 - \alpha}{1 - \beta}$ .

The normalized Gilbert-Varshamov distance  $\gamma_{GV}(R) \in [0, 1/2]$  is the value  $\gamma_{GV}(R)$  that gives  $D_B(\gamma_{GV}(R) \parallel 1/2) = R$ . The inverse of the normalized Gilbert-Varshamov distance is denoted  $\gamma_{GV}^{-1}(\alpha)$  for  $0 \leq \alpha \leq 1/2$ .

Bold face  $\mathbf{z}$  or  $\mathbf{z}_1^N$  denotes the length- $N$  binary sequence  $(z_1 z_2 \dots z_N)$  where  $z_t$  is the  $t$ -th bit of  $\mathbf{z}$ . The Hamming weight of  $\mathbf{z}$  is equal to the cardinality of the set  $\{t \in [1 : N] : z_t = 1\}$  and is denoted as  $w_H(\mathbf{z})$ . The bit-wise XOR operation is symbolized by  $\oplus$  and the bit-wise XOR of two binary sequences  $\mathbf{x}$  and  $\mathbf{y}$  is written as  $\mathbf{x} \oplus \mathbf{y}$ . The Hamming distance between two binary sequences  $\mathbf{x}$  and  $\mathbf{y}$  is the cardinality of the set  $\{t \in [1 : N] : x_t \neq y_t\}$  and is denoted as  $d_H(\mathbf{x}, \mathbf{y}) := |\{t \in [1 : N] : x_t \neq y_t\}|$ .

We will use the notation  $\doteq$ ,  $\dot{\leq}$ , and  $\dot{\geq}$  as follows: 1)  $a_N \doteq e^{Nd}$  denotes  $d = \liminf_{N \rightarrow \infty} \frac{\ln a_N}{N}$ . 2)  $a_N \dot{\leq} e^{Nd}$  denotes  $d \geq \liminf_{N \rightarrow \infty} \frac{\ln a_N}{N}$ . 3)  $a_N \dot{\geq} e^{Nd}$  denotes  $d \leq \liminf_{N \rightarrow \infty} \frac{\ln a_N}{N}$ .

## II. PROBLEM STATEMENT: NOISY 20 QUESTIONS FOR ESTIMATION OF A TARGET VARIABLE

We consider the following state estimation problem in the context of a noisy 20 questions game between a player and an oracle who communicate over a channel. The objective of the player is to estimate the value of a target variable, or state,  $X \sim \text{unif}[0, 1]$  by posing a sequence of binary queries to the oracle and receiving noisy answers. To estimate  $X$ , the player asks the oracle whether  $X$  is located within some sub-region  $Q \subset [0, 1]$ , which may be connected or non-connected, and receives a noisy binary answer  $Y \in \{0, 1\}$  based on the correct answer  $Z(X) = \mathbb{1}(X \in Q)$  with error probability  $\epsilon \in [0, 1/2]$ . The oracle always provides a correct binary answer  $Z(X) = \mathbb{1}(X \in Q)$  to the player's query, and the channel through which the oracle's binary answer is transmitted to the player is modeled as a binary symmetric channel, BSC( $\epsilon$ ).

The player asks the sequence of  $N$  questions as a sequence of querying regions  $(Q_1, Q_2, \dots, Q_N)$ . The oracle provides correct answers  $(Z_1, Z_2, \dots, Z_N)$  to the queries about the target variable  $X$ , and the player receives the noisy version  $(Y_1, Y_2, \dots, Y_N)$  of the oracle's answers through  $N$  uses of the BSC( $\epsilon$ ). Based on these answers, the player calculates an estimate  $\hat{X}_N$  of  $X$ . For a given cost function  $c(x, \hat{x}_N)$  between the true value  $x$  and the estimate  $\hat{x}_N$ , the player's goal is to find the optimal sequence of querying regions  $(Q_1, Q_2, \dots, Q_N)$  and the estimator  $\hat{X}_N(Y_1, \dots, Y_N)$  that minimize the expected cost function. That is, the player aims to achieve

$$\min_{(Q_1, Q_2, \dots, Q_N), \hat{X}_N(\cdot)} \mathbb{E}[c(X, \hat{X}_N)] \quad (1)$$

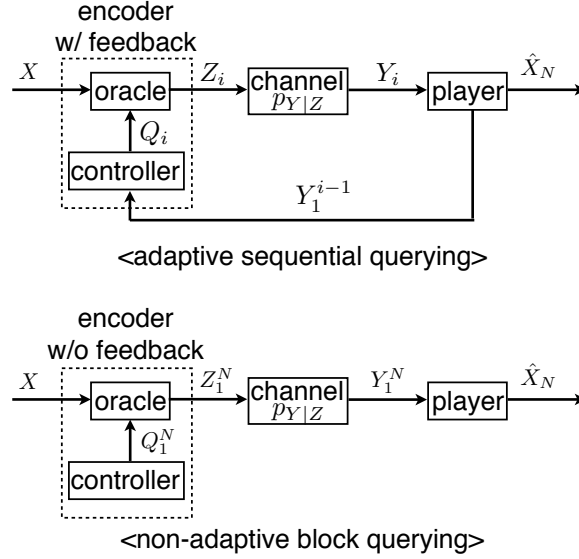


Fig. 1. Noisy 20 questions problem between an oracle and a player over a BSC( $\epsilon$ ). The controller generates questions using feedback (adaptive sequential querying) or operating open-loop (non-adaptive block querying). For adaptive sequential querying, the controller generates queries  $Q_i$  adaptively based on past answers  $Y_1^{i-1}$ , whereas for non-adaptive block querying the controller generates a length- $N$  block of queries  $Q_1^N = (Q_1, \dots, Q_N)$  non-adaptively as a single block. The oracle gives the correct answer  $Z_i$  to the query  $Q_i$  about the target variable  $X$ . The player receives a noisy version  $Y_i$  of the oracle's answer  $Z_i$  through a BSC( $\epsilon$ ), and outputs an estimate  $\hat{X}_N$  after receiving  $Y_1^N = (Y_1, \dots, Y_N)$ .

where the expectation is taken over the joint distribution of  $(X, Y_1, Y_2, \dots, Y_N)$ . Note that the joint distribution of  $(X, Y_1, Y_2, \dots, Y_N)$  depends on the querying regions  $(Q_1, Q_2, \dots, Q_N)$ .

The sequence of questions is designed by a controller that may either use feedback (adaptive sequential querying) or operate open-loop (non-adaptive block querying) as depicted in Fig. 1. Depending on whether the questions are designed with or without the benefit of feedback, the optimum querying strategy and the corresponding performance can vary. In the next section, we highlight differences between adaptive sequential querying and non-adaptive block querying and show a connection between the noisy 20 questions problem and the channel coding problem.

#### A. Adaptive vs. Non-adaptive Querying Strategies and Associated Channel Coding Problems

In the adaptive case the  $i$ -th querying region  $Q_i$  can be updated based on past answers  $Y_1^{i-1} := (Y_1, \dots, Y_{i-1})$  to previous queries. For this case, the controller uses the updated posterior distribution  $p(x|y_1^{i-1})$  of  $X$  to design the next query, i.e., the region  $Q_i$ . For example, the  $i$ -th querying region  $Q_i$  can be designed to equalize the probabilities of  $X$  belonging to  $Q_i$  and of  $X$  not belonging to  $Q_i$ , respectively, for given collected answers  $Y_1^{i-1} = y_1^{i-1}$ :

$$\Pr(X \in Q_i | Y_1^{i-1} = y_1^{i-1}) = \Pr(X \notin Q_i | Y_1^{i-1} = y_1^{i-1}) = 1/2. \quad (2)$$

Since the channel input (the oracle's binary answer)  $Z_i(X)$  is an indicator random variable of the event  $\{X \in Q_i\}$ , for the choice of  $Q_i$  satisfying (2) the corresponding channel input  $Z_i$  has the optimum input distribution for the

BSC( $\epsilon$ )

$$\Pr(Z_i = 0 | Y_1^{i-1} = y_1^{i-1}) = \Pr(Z_i = 1 | Y_1^{i-1} = y_1^{i-1}) = 1/2, \quad (3)$$

and thus it achieves the maximum mutual information of the BSC( $\epsilon$ ) given any collected answers  $y_1^{i-1}$ . Specifically, the corresponding conditional mutual information between  $Z_i$  and  $Y_i$  given  $Y_1^{i-1} = y_1^{i-1}$  is

$$I(Z_i; Y_i | Y_1^{i-1} = y_1^{i-1}) = C \quad (4)$$

where

$$C := \ln 2 - (-\epsilon \ln \epsilon - (1 - \epsilon) \ln(1 - \epsilon)). \quad (5)$$

To summarize, in adaptive sequential querying the channel input  $Z_i(X) = \mathbb{1}(X \in Q_i)$  depends on the previous channel outputs  $Y_1^{i-1}$ , since the querying region  $Q_i$  depends on  $Y_1^{i-1}$ . As depicted in the upper figure of Fig. 1, the combined operation of the controller and the oracle can be thought of as an encoder in a feedback communication system. Therefore, there is a one-to-one mapping between designing an adaptive sequential querying strategy and designing a sequential channel coder with noiseless feedback.

In the non-adaptive case the querying regions  $Q_1^N := (Q_1, \dots, Q_N)$  are specified in advance, before observing any of the answers from the oracle. Assume that the controller generates queries on the first  $k$  bits in the dyadic expansion of  $X \approx 0.B_1 \dots B_k$ ,  $B_i \in \{0, 1\}$  for  $i = 1, \dots, k$ . The resolution parameter  $k$  may depend on the number of queries  $N$ . Discovering  $(B_1, \dots, B_k)$  is equivalent to finding the index  $M = \sum_{i=1}^k B_i 2^{k-i} \in \{0, \dots, 2^k - 1\}$  of the interval  $I_M := [M2^{-k}, (M+1)2^{-k})$  that contains  $X$ . Here the domain  $[0, 1]$  of  $X$  is uniformly quantized into  $2^k$  disjoint sub-intervals  $\{I_0, \dots, I_{2^k-1}\}$  of length  $2^{-k}$ . If the oracle's answer  $Z_i$  to the question  $Q_i$  can be transmitted to the player without noise, i.e.,  $\epsilon = 0$ , then by querying each coefficient of the dyadic expansion of  $X$ , from the MSB to the LSB, the player can discover the  $N$  most significant bits  $(B_1, \dots, B_N)$  of  $X$  without error. However, in the case of a noisy channel, the player needs to ask redundant questions in order to accurately estimate the  $k$  most significant bits of  $X$  for some  $k < N$ .

Non-adaptive block querying can be mapped to an equivalent problem of length- $N$  block channel coding over a BSC( $\epsilon$ ). The rate of the block code is defined as  $R = (k \ln 2)/N$  (nats/channel use) for the resolution of  $k$  bits of  $X$ . Designing a block of questions  $(Q_1, \dots, Q_N)$  to discover the index  $M$  of the sub-interval  $I_M$  containing  $X$  can be thought of as designing a length- $N$  and rate- $R$  block code, or, more specifically, defining an encoding map  $f : \{0, \dots, 2^k - 1\} \rightarrow \{0, 1\}^N$ , to reliably transmit one of the  $2^k$  messages through  $N$  uses of the channel with channel coding rate  $R = (k \ln 2)/N$ .

A block of questions specifies the encoding map  $f : \{1, \dots, 2^k\} \rightarrow \{0, 1\}^N$ , and vice versa. The one-to-one mapping between the two is described as follows. Define sub-intervals  $I_m := [m2^{-k}, (m+1)2^{-k})$  for  $m \in \{0, \dots, 2^k - 1\}$ . We restrict the querying region  $Q_i$  to be the union of a subset of the quantized intervals  $\{I_0, \dots, I_{2^k-1}\}$ . In other words, we fix the maximum resolution of the querying interval as  $2^{-k}$ . Let  $z_i^{(m)}$  denote the  $i$ -th bit of the codeword  $f(m) = (z_1^{(m)}, \dots, z_N^{(m)})$  for a message  $m$  given an encoding map  $f : \{0, \dots, 2^k - 1\} \rightarrow \{0, 1\}^N$ . Note that the bit  $z_i^{(m)}$  is the oracle's binary answer to the query  $Q_i$  indicating

whether  $x \in Q_i$  for  $x \in I_m$ . Therefore, the bit  $z_i^{(m)}$  equals 1 if and only if  $I_m \subset Q_i$ , i.e.,

$$z_i^{(m)} = \mathbb{1}(I_m \subset Q_i). \quad (6)$$

On the other hand, when the encoding map  $f(\cdot)$  is specified, the associated  $i$ -th querying region  $Q_i$  becomes the union of the sub-intervals  $\{I_{m'}\}$  for message  $m'$ 's such that the  $i$ -th answer bit  $z_i^{(m')}$  equals 1, i.e.,

$$Q_i = \bigcup_{\{m': z_i^{(m')}=1\}} I_{m'}. \quad (7)$$

Given the block of questions  $(Q_1, \dots, Q_N)$ , the oracle transmits the length- $N$  binary answer bits  $f(m)$  when  $x \in I_m$  through  $N$  uses of the BSC( $\epsilon$ ), and the player tries to decode the message  $m$  given a noisy version of the codeword.

Thus both adaptive sequential querying and non-adaptive block querying can be mapped to associated channel coding problems in information transmission through a noisy channel, with and without feedback, respectively. However, different from information transmission problems where the goal is to achieve reliable communication at some positive rate  $0 < R \leq C$ , the objective of the noisy 20 questions problem for state estimation is to minimize estimation error  $\mathbb{E}[c(X, \hat{X}_N)]$ . In the next section, we introduce two different types of estimation errors that will be considered in this paper and discuss what kind of properties are desired for channel coding to minimize these estimation errors.

### B. Estimation Errors: Mean Squared Error and Quantized Mean Squared Error

We consider two types of estimation error. The first is the mean squared error (MSE)  $\mathbb{E}[|X - \hat{X}_N|^2]$  where  $\hat{X}_N$  is the estimate of  $X$  after  $N$  queries. The second is the quantized MSE  $\mathbb{E}[c_q(X, \hat{X}_N)]$  where the quantized cost function  $c_q(X, \hat{X}_N)$  with  $2^k$  levels is a stepwise function defined as

$$c_q(X, \hat{X}_N) = \begin{cases} 0, & 0 \leq |X - \hat{X}_N| \leq \frac{2^{-k}}{2}, \\ (d2^{-k})^2, & d2^{-k} - \frac{2^{-k}}{2} < |X - \hat{X}_N| \leq d2^{-k} + \frac{2^{-k}}{2} \text{ for } d \in \{1, \dots, 2^k - 2\}, \\ ((2^k - 1)2^{-k})^2, & (2^k - 1)2^{-k} - \frac{2^{-k}}{2} < |X - \hat{X}_N| \leq 1, \end{cases} \quad (8)$$

for  $X, \hat{X}_N \in [0, 1]$ . We consider this cost function when the objective of the problem is to estimate the value of  $X$  up to the first  $k$  bits  $(B_1, \dots, B_k)$  in the dyadic expansion of  $X$ .

Let  $(\hat{B}_1, \dots, \hat{B}_k)$  denote the estimate of  $(B_1, \dots, B_k)$  and  $\hat{M} = \sum_{i=1}^k \hat{B}_i 2^{k-i}$  denote the estimate of the message  $M = \sum_{i=1}^k B_i 2^{k-i}$ . We define the decoding error distance  $d(M, \hat{M})$  between  $M$  and  $\hat{M}$  as

$$d(M, \hat{M}) := |M - \hat{M}| = \left| \sum_{i=1}^k (B_i - \hat{B}_i) 2^{k-i} \right|. \quad (9)$$

By defining the finite resolution estimator  $\hat{X}_{N, \text{finite}}$  as

$$\hat{X}_{N, \text{finite}} := \hat{M} 2^{-k} + 2^{-k}/2, \quad (10)$$

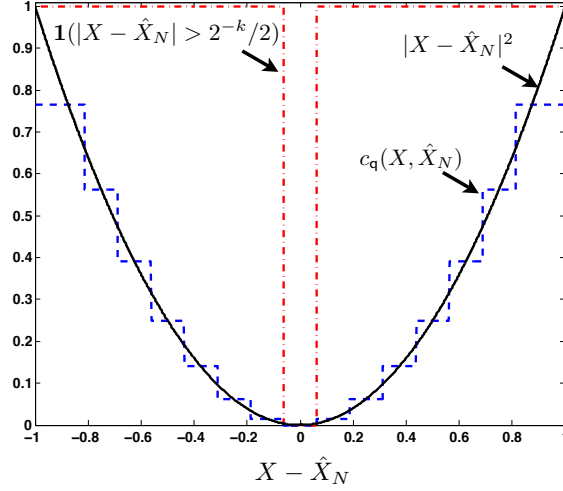


Fig. 2. Plot of three different cost functions:  $|X - \hat{X}_N|^2$  for the MSE,  $c_q(X, \hat{X}_N)$  for the quantized MSE and  $\mathbb{1}(\hat{M} \neq M) = \mathbb{1}(|X - \hat{X}_N| > 2^{-k}/2)$  for the block decoding error probability when the resolution parameter  $k = 3$ .

the quantized MSE  $c_q(X, \hat{X}_N)$  with  $\hat{X}_N = \hat{X}_{N,\text{finite}}$  can be written in terms of the decoding error distance as

$$c_q(X, \hat{X}_{N,\text{finite}}) = 2^{-2k} (d(M, \hat{M}))^2. \quad (11)$$

Note that the quantized MSE equals 0 when the player correctly decodes the message  $M$ . The error increases proportionally to the square of the decoding error distance. On the other hand, in information transmission problems where the cost function is  $\mathbb{1}(\hat{M} \neq M)$ , decoding error is claimed when  $\hat{M} \neq M$ , i.e., when  $d(M, \hat{M}) \neq 0$ , and the cost of incorrect decoding is the same for every  $\hat{M} \neq M$  regardless of the decoding error distance  $d(M, \hat{M})$ . This difference in the cost functions makes the desired channel coding strategy for state estimation problem different from that of the information transmission problem. Fig. 2 shows the three different cost functions,  $|X - \hat{X}_N|^2$  for the MSE,  $c_q(X, \hat{X}_N)$  for the quantized MSE and  $\mathbb{1}(\hat{M} \neq M) = \mathbb{1}(|X - \hat{X}_N| > 2^{-k}/2)$  for the block decoding error probability when the resolution parameter  $k = 3$ .

The quantized MSE  $\mathbb{E}[c_q(X, \hat{X}_N)]$  approximates the MSE  $\mathbb{E}[|X - \hat{X}_N|^2]$ . In particular, with the finite resolution estimator  $\hat{X}_N = \hat{X}_{N,\text{finite}}$ , the MSE  $\mathbb{E}[|X - \hat{X}_{N,\text{finite}}|^2]$  can be written as a sum of the quantized MSE  $\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})]$  and the error from finite resolution,

$$\mathbb{E}[|X - \hat{X}_{N,\text{finite}}|^2] = \mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})] + c2^{-2k}, \quad (12)$$

for some constant  $0 < c \leq 1/4$ . This can be shown by writing the difference between the two expected errors as a sum of errors conditioned on the decoding error distance  $d(M, \hat{M}) = d$  for  $d \in \{0, \dots, 2^k - 1\}$ ,

$$\begin{aligned} & \mathbb{E}[|X - \hat{X}_{N,\text{finite}}|^2] - \mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})] \\ &= \sum_{d=0}^{2^k-1} \Pr(d(M, \hat{M}) = d) \mathbb{E} \left[ \left( |X - \hat{X}_{N,\text{finite}}|^2 - d^2 2^{-2k} \right) \middle| d(M, \hat{M}) = d \right]. \end{aligned} \quad (13)$$



For  $d = 0$ , the conditional expectation in (13) is bounded above by  $2^{-2k}/4$ . Given that  $X \sim \text{unif}[0, 1]$ , conditioned on  $d(M, \hat{M}) = d$  for  $d \in \{1, \dots, 2^k - 1\}$ ,  $|X - \hat{X}_{N, \text{finite}}|$  is uniformly distributed over  $[d2^{-k} - 2^{-k}/2, d2^{-k} + 2^{-k}/2]$ .

Thus

$$\mathbb{E} \left[ \left( |X - \hat{X}_{N, \text{finite}}|^2 - d^2 2^{-2k} \right) | d(M, \hat{M}) = d \right] = \frac{1}{12} 2^{-2k}. \quad (14)$$

Therefore, the difference between the MSE and quantized MSE is bounded above by a scale factor of  $2^{-2k}$  as in (12).

Consider the case when the resolution  $k$  bits of the estimator  $\hat{X}_{N, \text{finite}}$  increases linearly in the number of queries  $N$  as  $k = NR/\ln 2$  for some fixed positive rate  $R > 0$ . Let  $E_{\text{MSE}, \text{policy}}^*(R)$  and  $E_{\text{q}, \text{policy}}^*(R)$  denote the best achievable exponentially decreasing rates of the MSE and the quantized MSE in  $N$  at a fixed rate  $R$ , respectively, for some policy, i.e.,

$$E_{\text{MSE}, \text{policy}}^*(R) := \liminf_{N \rightarrow \infty} \frac{-\ln \mathbb{E}[|X - \hat{X}_{N, \text{finite}}|^2]}{N}, \quad (15)$$

$$E_{\text{q}, \text{policy}}^*(R) := \liminf_{N \rightarrow \infty} \frac{-\ln \mathbb{E}[c_{\text{q}}(X, \hat{X}_{N, \text{finite}})]}{N}. \quad (16)$$

Then the equality in (12) implies that for large  $N$ , the exponential convergence rate of the MSE  $\mathbb{E}[|X - \hat{X}_{N, \text{finite}}|^2]$  in  $N$  is dominated by the minimum of the rate of the quantized MSE and  $2R$ , i.e.,

$$E_{\text{MSE}, \text{policy}}^*(R) = \min\{E_{\text{q}, \text{policy}}^*(R), 2R\}. \quad (17)$$

For sufficiently large  $R > 0$  where  $E_{\text{q}, \text{policy}}^*(R) \leq 2R$ , the MSE exponent is identical to the quantized MSE exponent. In this paper, we analyze performance of a querying policy by calculating the best achievable quantized MSE exponent  $E_{\text{q}, \text{policy}}^*(R)$  at a fixed rate  $R > 0$  for querying resolution of  $k = NR/\ln 2$  bits. Once the quantized MSE exponent is calculated for every  $R > 0$ , by using (17) we can also calculate the resulting MSE exponent.

We next show how the MSE and the quantized MSE can be bounded below and above in terms of the block decoding error events  $\{\hat{M} \neq M\}$  or bit decoding error events  $\{\hat{B}_i \neq B_i\}$ ,  $i \in \{1, \dots, k\}$ . The block decoding error  $\{\hat{M} \neq M\}$  occurs when any of  $B_i$ 's are incorrectly decoded. For a given cost function  $c(x, \hat{x}_N)$ , the expected estimation error can be written in terms of block decoding events  $\{\hat{M} = M\}$  and  $\{\hat{M} \neq M\}$  as

$$\mathbb{E}[c(X, \hat{X}_N)] = \Pr(\hat{M} \neq M) \mathbb{E}[c(X, \hat{X}_N) | \hat{M} \neq M] + \Pr(\hat{M} = M) \mathbb{E}[c(X, \hat{X}_N) | \hat{M} = M].$$

With the finite resolution estimator  $\hat{X}_{N, \text{finite}} = \hat{M}e^{-NR} + e^{-NR}/2$ , the MSE and the quantized MSE can be bounded above as

$$\mathbb{E}[|X - \hat{X}_{N, \text{finite}}|^2] \leq \Pr(\hat{M} \neq M) + (e^{-NR}/2)^2, \quad (18)$$

$$\mathbb{E}[c_{\text{q}}(X, \hat{X}_{N, \text{finite}})] \leq \Pr(\hat{M} \neq M), \quad (19)$$

by using  $\mathbb{E}[|X - \hat{X}_{N, \text{finite}}|^2 | \hat{M} = M] \leq (e^{-NR}/2)^2$  and  $\mathbb{E}[c_{\text{q}}(X, \hat{X}_{N, \text{finite}}) | \hat{M} = M] = 0$ , respectively. The achievable exponent of the MSE in (18) is determined by the trade-off between the exponentially decreasing rate of  $\Pr(\hat{M} \neq M)$  at a fixed rate  $R$  and the exponent  $2R$ . On the other hand, in (19), the achievable exponent of the quantized MSE is determined by the exponentially decreasing rate of  $\Pr(\hat{M} \neq M)$  in  $N$  at a fixed rate  $R$ .

Tighter bounds on the two estimation errors can be found by expanding the errors in terms of the bit error probabilities. For a cost function  $c(x, \hat{x}_N)$ , the expected cost can be written as

$$\begin{aligned} \mathbb{E}[c(X, \hat{X}_N)] &= \sum_{i=1}^k \Pr(\hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}) \mathbb{E}[c(X, \hat{X}_N) | \hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}] \\ &\quad + \Pr(\hat{B}_1^k = B_1^k) \mathbb{E}[c(X, \hat{X}_N) | \hat{B}_1^k = B_1^k] \end{aligned} \quad (20)$$

where the number of information bits is  $k = NR/\ln 2$ . Note that conditioned on the event  $\{\hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}\}$ , the cost functions  $|X - \hat{X}_{N,\text{finite}}|^2$  and  $c_q(X, \hat{X}_{N,\text{finite}})$  are bounded below and above

$$\begin{aligned} \left| X - \sum_{t=1}^i B_t 2^{-t} \right|^2 &\leq |X - \hat{X}_{N,\text{finite}}|^2 \leq 2^{-2(i-1)}, \\ 2^{-2k} \left( d \left( M, \sum_{t=1}^i B_t 2^{k-t} - 1 \right) \right)^2 &\leq c_q(X, \hat{X}_{N,\text{finite}}) \leq 2^{-2(i-1)}. \end{aligned} \quad (21)$$

Again, for  $X \sim \text{unif}[0, 1]$ , conditioned on the event  $\{\hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}\}$ ,  $|X - \sum_{t=1}^i B_t 2^{-t}|$  is uniformly distributed over  $[0, 2^{-i}]$  and the decoding error distance  $d \left( M, \sum_{t=1}^i B_t 2^{k-t} - 1 \right)$  is uniformly distributed over the discrete set  $\{1, \dots, 2^{k-i}\}$ . Therefore,

$$\begin{aligned} \frac{2^{-2i}}{3} &\leq \mathbb{E}[|X - \hat{X}_{N,\text{finite}}|^2 | \hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}] \leq 2^{-2(i-1)}, \\ \frac{2^{-2i}}{3} &\leq \mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}}) | \hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}] \leq 2^{-2(i-1)}. \end{aligned} \quad (22)$$

By using these bounds and (20), we can bound the MSE below and above as,

$$\begin{aligned} \sum_{i=1}^k \Pr(\hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}) \frac{2^{-2i}}{3} + \Pr(\hat{B}_1^k = B_1^k) \frac{2^{-2k}}{12} &\leq \mathbb{E}[|X - \hat{X}_{N,\text{finite}}|^2] \\ &\leq \sum_{i=1}^k \Pr(\hat{B}_i \neq B_i) 2^{-2(i-1)} + 2^{-2k}, \end{aligned} \quad (23)$$

and the quantized MSE as

$$\sum_{i=1}^k \Pr(\hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}) \frac{2^{-2i}}{3} \leq \mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})] \leq \sum_{i=1}^k \Pr(\hat{B}_i \neq B_i) 2^{-2(i-1)}. \quad (24)$$

These bounds show how each bit error probability contributes to the average cost of estimation errors. In the upper bounds on the MSE (23) and on the quantized MSE (24), we can see that the weights on the bit error probabilities decrease exponentially in  $i$  as the bit position  $i$  increases corresponding to lower significance. In order to minimize the upper bounds on the MSE and on the quantized MSE for a fixed number of querying  $N$ , we need to design a querying strategy (or the associated channel coding) that can provide unequal error protection depending on the bit positions. This property differentiates a good channel coding strategy for state estimation from that for information transmission. In classical information transmission problems where the cost function is  $\mathbb{1}(\hat{M} \neq M)$ , any bit error event  $\{\hat{B}_i \neq B_i\}$  results in the same cost. Therefore, the optimal coding strategy to minimize the decoding error probability  $\Pr(\hat{M} \neq M) = \mathbb{E}[\mathbb{1}(\hat{M} \neq M)]$  at a fixed rate  $R$  provides uniform error protection for all the information bits. On the other hand, in a state estimation problem, the optimal coding strategy should provide unequal error protection.

### III. REVIEW OF THREE DIFFERENT QUERYING STRATEGIES

In this section, we review three well-known querying policies including the adaptive bisection policy [12], the non-adaptive UEP repetition policy [1], and the non-adaptive block querying policy based on random block coding [14] in Sections III-A, III-B, and III-C, respectively. The performance of these policies are analyzed by the best achievable quantized MSE exponent defined in (16) with the finite resolution estimator  $\hat{X}_{N,\text{finite}}$  (10).

#### A. Adaptive Bisection Policy

For adaptive sequential querying, greedy successive entropy minimization of a target variable is often proposed as a way to design a querying strategy for estimation of the target variable [7], [9]. Successive entropy minimization strategies select the binary query that maximally reduces the remaining uncertainty of the target variable at each round. This can be accomplished by choosing a querying region  $Q_i$  that balances the probability of the event  $\{X \in Q_i\}$  and the probability of the event  $\{X \notin Q_i\}$ , given past answers  $y_1^{i-1}$ , i.e.,

$$\Pr(X \in Q_i | Y_1^{i-1} = y_1^{i-1}) = \Pr(X \notin Q_i | Y_1^{i-1} = y_1^{i-1}) = 1/2. \quad (25)$$

The uncertainty of the target variable is quantified by the differential entropy  $h(X) := -\int p(x) \ln p(x) dx$  where  $X \sim p(x)$ , and the reduction of the uncertainty by the  $i$ -querying equals

$$h(X | Y_1^{i-1} = y_1^{i-1}) - h(X | Y_i, Y_1^{i-1} = y_1^{i-1}) \quad (26)$$

where  $Y_i$  is the noisy observation of the oracle's answer  $Z_i = \mathbb{1}(X \in Q_i)$  transmitted through the BSC( $\epsilon$ ). For  $Q_i$  satisfying (25), the reduction of the conditional entropy in (26) is equal to the channel capacity  $C := \max_{Y_i} I(X; Y_i | Y_1^{i-1} = y_1^{i-1}) = H_B(1/2) - H_B(\epsilon)$  where  $I(X; Y_i | Y_1^{i-1} = y_1^{i-1})$  is the conditional mutual information between  $X$  and  $Y_i$  given  $Y_1^{i-1} = y_1^{i-1}$ . After  $N$  rounds of querying, successive entropy minimization strategies reduce the entropy of  $X$  by  $NC$ . From the elementary bound

$$h(X) - h(X | Y_1^N) \leq \max_{Y_1^N} I(X; Y_1^N) = NC, \quad (27)$$

we can see that the successive entropy minimization policy achieves the maximum possible entropy reduction of  $X$  among policies that make  $N$ -uses of the BSC( $\epsilon$ ).

The bisection policy, which is also called Horstein's coding scheme [12], is one example of a successive entropy minimization policy. This policy asks whether  $X$  lies to the left or right of the median of the posterior distribution, which is updated based on past answers. The left figure of Fig. 3 illustrates the bisection policy for the first two rounds of querying. At the first round, the value of  $X$  is uniformly distributed over  $[0, 1]$  and the median of the prior distribution equals  $1/2$ . Thus, the player asks whether  $X$  belongs to the right half of the region of interest  $[0, 1]$  by choosing  $Q_1 = [1/2, 1]$ , i.e., the player tries to extract the most significant bit of  $X$ . Given the observed answer  $Y_1 \in \{0, 1\}$ , the player updates the posterior distribution  $p(x|y_1)$  of  $X$ , and then chooses  $Q_2 \subset [0, 1]$  that bisects the posterior distribution such that  $\Pr(X \in Q_2 | Y_1 = y_1) = 1/2$ , i.e., it queries whether  $X$  lies to the right of the median of the posterior distribution  $p(x|y_1)$ . Depending on the answer  $Y_1$  of the previous query, the updated

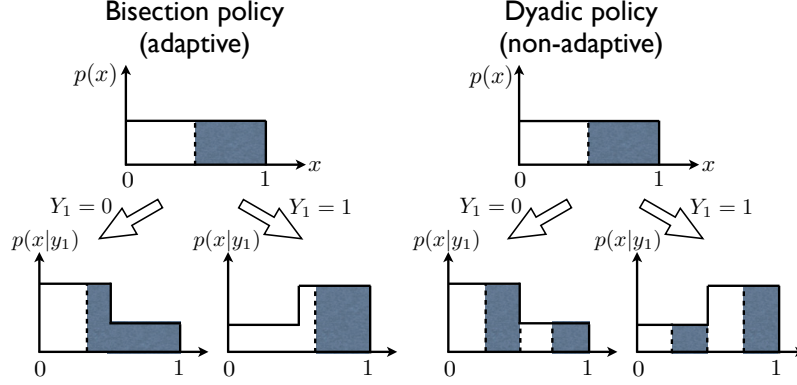


Fig. 3. Illustration of two successive entropy minimization policies, adaptive bisection policy and non-adaptive dyadic policy, for the first two rounds. The shaded regions correspond to the posterior distribution over the querying region  $Q_i$  at the  $i$ -th round. The bisection policy designs  $Q_i$  to be right region of the median of the posterior distribution, while the dyadic policy assigns  $Q_i$  to be the region that corresponds to the  $i$ -th bit  $B_i$  of the binary expansion of  $X$  equals 1. The querying region  $Q_i$  of the bisection policy changes depending on the received answers  $Y_1^{i-1}$  of the previous queries, while that of the dyadic policy does not change on  $Y_1^{i-1}$ . For both the querying strategies, the shaded areas take  $1/2$  of the posterior distribution.

posterior distribution  $p(x|y_1)$  and the median of the distribution change, so that the second querying region  $Q_2$  changes as a function of the answer of the previous query. At each round, the player keeps updating the posterior distribution  $p(x|y_1^{i-1})$  of the target variable given collected answers and designs the querying region  $Q_i$  to be right of the median of the updated  $p(x|y_1^{i-1})$ .

The bisection policy is known to work well in practice, but there are few available theoretical guarantees for the performance of this policy. Here we demonstrate that among successive entropy minimization policies satisfying (25) for every  $i \in \{1, \dots, N\}$  the bisection policy maximally reduces the conditional variance of  $X$  at each round. More specially, we show that the bisection policy chooses  $Q_i$  that maximizes the predicted variance reduction at the  $i$ -th round given the answers  $y_1^{i-1}$  of the previous rounds, i.e.,

$$\max_{Q_i} (\text{Var}(X|Y_1^{i-1} = y_1^{i-1}) - \mathbb{E}[\text{Var}(X|Y_i, Y_1^{i-1} = y_1^{i-1})]). \quad (28)$$

The predicted variance reduction depends on the choice of the querying region  $Q_i$  since the posterior distribution  $p(x|y_1^i)$  is a function of  $Q_i$ . The minimum mean square error (MMSE) estimator  $\hat{X}_{N, \text{MMSE}} = \mathbb{E}[X|Y_1^N = y_1^N]$  minimizes the MSE and make it equal to the conditional variance of  $X$  given  $Y_1^N$ ,

$$\min_{\hat{X}_N} \mathbb{E}[|X - \hat{X}_N|^2] = \mathbb{E}[(X - \hat{X}_{N, \text{MMSE}})^2] = \mathbb{E}[\text{Var}(X|Y_1^N = y_1^N)]. \quad (29)$$

Therefore, the bisection policy, which maximizes the predicted variance reduction in (28), is the MSE-optimal myopic (greedy) policy.

*Proposition 1: Among successive entropy minimization policies, which choose the  $i$ -th querying regions  $Q_i$  satisfying  $\Pr(X \in Q_i|Y_1^{i-1} = y_1^{i-1}) = \Pr(X \notin Q_i|Y_1^{i-1} = y_1^{i-1}) = 1/2$  given previous answers  $y_1^{i-1}$ , the bisection policy maximizes the predicted one-step variance reduction (28) at each round.*

*Remark 1:* In the proof of Proposition 1 in Appendix A, we show that the predicted variance reduction of  $X$  due to the  $i$ -th query  $Q_i$  is a function of not only  $\{\Pr(X \in Q_i|Y_1^{i-1} = y_1^{i-1}), \Pr(X \notin Q_i|Y_1^{i-1} = y_1^{i-1})\}$  but also  $\{\mathbb{E}[X \in Q_i|Y_1^{i-1} = y_1^{i-1}], \mathbb{E}[X \notin Q_i|Y_1^{i-1} = y_1^{i-1}]\}$ . Successive entropy minimization policies select  $Q_i$  that makes  $\Pr(X \in Q_i|Y_1^{i-1} = y_1^{i-1}) = 1/2$  but do not care about the corresponding  $\{\mathbb{E}[X \in Q_i|Y_1^{i-1} = y_1^{i-1}], \mathbb{E}[X \notin Q_i|Y_1^{i-1} = y_1^{i-1}]\}$ , which also governs the conditional variance of  $X$ . What we show in Proposition 1 is that the bisection policy selects  $Q_i$  whose associated  $\{\mathbb{E}[X \in Q_i|Y_1^{i-1} = y_1^{i-1}], \mathbb{E}[X \notin Q_i|Y_1^{i-1} = y_1^{i-1}]\}$  maximizes the predicted variance reduction among all choices of  $Q_i$  satisfying  $\Pr(X \in Q_i|Y_1^{i-1} = y_1^{i-1}) = 1/2$ . More discussion on successive entropy minimization policies and the proof of Proposition 1 will be provided in Appendix A.

In this paper, we are particularly interested in the error rates of convergence achievable with the bisection policy. Even though the error rate for the bisection policy is very hard to analyze and not known in general, a slight modification of the bisection policy proposed by Burnashev and Zigangirov in [11], and called the BZ algorithm, is analyzable. The BZ algorithm works very similarly to the bisection policy, except that the boundary of the querying regions is not equal to the median of the posterior distribution. Rather, the BZ boundary is chosen among a set of uniformly quantized thresholds  $\mathcal{T} = \{0, 2^{-k}, 2(2^{-k}), \dots, 2^k(2^{-k})\}$  with resolution  $2^{-k}$ . More specifically, the threshold is chosen by sampling between the two points in the set  $\mathcal{T}$  that are closest to the median of the posterior distribution. After  $N$  rounds of querying with the BZ algorithm, the controller finds the sub-interval  $I_{\hat{m}} = [\hat{m}2^{-k}, (\hat{m} + 1)2^{-k})$  that maximizes the posterior probability of  $X$ . When  $M$  is the true index of the interval  $I_M = [M2^{-k}, (M + 1)2^{-k})$  where the target variable  $X$  belongs, the probability of the event  $\{\hat{M} \neq M\}$  with the BZ algorithm decreases exponentially in  $N$  as

$$\Pr(\hat{M} \neq M) \leq e^{-N(-\ln(1/2 + \sqrt{\epsilon(1-\epsilon)}) - R)} \quad (30)$$

where  $R = (k \ln 2)/N$  [11], [4]. Since the quantized MSE can be bounded above by the block decoding error as shown in (19), the quantized MSE exponent defined in (16) is bounded below by the exponent on the right hand side of (30).

*Lemma 1 (Quantized MSE exponent with BZ algorithm [11]):* The best achievable quantized MSE exponent with the BZ algorithm, denoted  $E_{q,BZ}^*(R)$ , is bounded below as

$$E_{q,BZ}^*(R) \geq E_{q,BZ}(R) := -\ln(1/2 + \sqrt{\epsilon(1-\epsilon)}) - R \quad (31)$$

when the resolution of the querying region is  $k = NR/\ln 2$  bits.

We will compare the achievable quantized MSE exponent  $E_{q,BZ}(R)$  with those of non-adaptive policies introduced in the following sections.

### B. Non-Adaptive Unequal Error Protection Repetition Querying

Different from the adaptive policy where the updated posterior distribution  $p(x|y_1^{i-1})$  is available to the controller for design of the  $i$ -th querying region  $Q_i$ , for the non-adaptive policy a block of queries is determined independently of previous answers from the oracle. Our objective is to design a block of queries to estimate  $X$  up to the first  $k$

bits in the binary expansion of  $X \approx 0.B_1 \dots B_k$  with the minimum estimation error  $\mathbb{E}[c(X, \hat{X}_N)]$  for a given cost function  $c(X, \hat{X}_N)$ .

We first point out that even for the non-adaptive case, there exists a block of queries  $(Q_1, \dots, Q_N)$  that does not depend on  $Y_1^N$  but still meets the condition (25) for successive entropy minimization policies for every  $Y_1^N \in \{0, 1\}^N$ . Such a policy is the dyadic policy [9] and works as follows: The dyadic policy queries the coefficients of the dyadic expansion of  $X \approx 0.B_1 \dots B_N$  from  $B_1$  to  $B_N$  one at a time over  $N$  rounds of querying. The right figure of Fig. 3 illustrates the procedure of the dyadic policy. At the first round, as in the bisection policy, it queries the MSB  $B_1$ . At the second round, regardless of  $Y_1 \in \{0, 1\}$  it queries the second MSB  $B_2$  by choosing  $Q_2$  to be  $Q_2 = [1/4, 2/4] \cup [3/4, 1]$ , which is the region of  $X$  corresponding to  $B_2 = 1$ . The player continues the procedure of asking about  $B_i$  at the  $i$ -th round for  $i = 1, \dots, N$ . Since the prior distribution of  $X$  is uniform over  $[0, 1]$ , the quantized bits  $\{B_i\}$  are i.i.d. with Bernoulli(1/2). Moreover, since the channel outputs  $Y_1^{i-1} \in \{0, 1\}^{i-1}$  contain information only about  $B_1^{i-1}$  but not about  $B_i$ , the events  $\{B_i = 1\}$  and  $\{B_i = 0\}$  are independent of  $Y_1^{i-1}$ . Therefore, the dyadic policy satisfies condition (25) for every  $y_1^{i-1} \in \{0, 1\}^{i-1}$  and achieves the maximum reduction (27) of the conditional entropy.

Even though the dyadic policy maximally reduces the uncertainty of  $X$  measured by the entropy, this policy fails to make the estimation error converge to 0 even when  $N \rightarrow \infty$ . This is because, in the BSC( $\epsilon$ ), with  $\epsilon \in (0, 1/2)$  probability the player receives an incorrect value for the information bit  $B_i$ . Since each bit  $B_i$  is queried only once by the dyadic policy, if the player receives an incorrect answer for some bit  $B_i$  there is no way to recover from this error. Therefore, the estimation error of the dyadic policy does not converge to 0.

To correctly estimate  $(B_1, \dots, B_k)$  through  $N$ -uses of the noisy BSC( $\epsilon$ ), the player needs to design a block of queries  $(Q_1, \dots, Q_N)$  with some redundancy, or equivalently design a block code with encoding map  $f : \{0, \dots, 2^k - 1\} \rightarrow \{0, 1\}^N$  to guarantee a reliable transmission of the information bits  $(B_1, \dots, B_k)$ . As pointed out earlier, the decoding error of each  $B_i$  has different effect on the estimation error. The different importances of  $B_i$ 's can be quantified by the different weights on the bit error probabilities  $\Pr(\hat{B}_i \neq B_i)$  in the upper bounds (23) and (24) on the MSE and on the quantized MSE, respectively. For non-adaptive block querying, in order to minimize the estimation error with a limited number  $N$  of queries it is desirable to provide different levels of error protection.

One way to provide unequal error protection is to repeat the query on the information bits multiple times, the number of repetitions varying in accordance with the desired level of error protection. Such a UEP repetition coding approach was considered in [1]. For this policy, the controller queries each of the bit  $B_i$  of  $X \approx B_1, B_2, \dots, B_k$  repeatedly  $N_i$  times and the oracle sends the uncoded bit  $B_i$  repeatedly by  $N_i$  uses of the BSC( $\epsilon$ ). The total number of channel uses is restricted to  $\sum_{i=1}^k N_i = N$  where  $k$  is the resolution of quantized  $X$ .

Note that this repetition coding policy cannot achieve the maximum entropy reduction (27) of the target variable  $X$ , which is achievable only when the player keeps asking the most informative query at each round. The repeated queries on  $B_i$  successively reduce the uncertainty of  $B_i$ , and the bit error probability of  $B_i$  decreases exponentially in the number of repeated queries,  $N_i$ . The minimum bit error probability of  $B_i$  is achievable with a simple majority voting algorithm, which claims the estimate  $\hat{B}_i$  to be the more frequently received binary value at the  $N_i$  channel

outputs. This simple algorithm is equivalent to maximum likelihood (ML) decoding for  $B_i$ .

*Lemma 2: When the oracle sends a binary bit  $B_i \sim \text{Bernoulli}(1/2)$  repeatedly  $N_i (\geq 1)$  times through a  $\text{BSC}(\epsilon)$ , the best achievable bit error probability with the majority voting algorithm decreases exponentially in  $N_i$  as*

$$\frac{e^{-1/(3N_i)}}{\sqrt{2\pi N_i}} e^{-N_i D_B(1/2\|\epsilon)} \leq \Pr(\hat{B}_i \neq B_i) \leq e^{-N_i D_B(1/2\|\epsilon)}. \quad (32)$$

*Proof:* Appendix B. ■

By assigning different numbers of repetitions  $(N_1, N_2, \dots, N_k)$  for each information bit  $B_i$  we can provide unequal error protection for the information bits. The remaining issue is the optimal solution for the number of repetitions  $(N_1, N_2, \dots, N_k)$  that minimize the estimation error where  $k$  is the total number of queried information bits. These should be selected to minimize the upper bound on the MSE  $\mathbb{E}[|X - \hat{X}_{N,\text{finite}}|^2]$  in (23) or the upper bound on the quantized MSE  $\mathbb{E}[c_q(X, \hat{X}_N)]$  in (24). Since the weight  $2^{-2(i-1)}$  on  $\Pr(\hat{B}_i \neq B_i)$  decreases exponentially in  $i$  and  $\Pr(\hat{B}_i \neq B_i)$  decreases exponentially in  $N_i$  as shown in (32), the optimum  $N_i^*$  that minimizes the upper bounds should decrease linearly in  $i$  from MSB to LSB. This condition then implies that  $N = \sum_{i=1}^k N_i^* = O(k^2)$ . Therefore, the maximum number of information bits that can be queried by repetition coding increases in  $N$  on the order of  $k = O(\sqrt{N})$ , and the corresponding rate  $R = k/N$  goes to 0 as  $N \rightarrow \infty$ . The resulting MSE and quantized MSE decrease exponentially only as  $\sqrt{N}$ . By using the lower bounds on the MSE in (23) and on the quantized MSE in (24), with the lower bound on  $\Pr(\hat{B}_i \neq B_i)$  in (32) we can show that this convergence rate is indeed tight.

By using the similar arguments, in [1] it was shown that with the UEP repetition code, the MSE minimized over all choices of  $(N_1, \dots, N_k)$  and  $k$  decreases exponentially in  $\sqrt{N}$  but not faster than that

$$c_1 e^{-c_2 \sqrt{N}} \leq \min_{(N_1, \dots, N_k), k} \mathbb{E}[|X - \hat{X}_{N,\text{finite}}|^2] \leq c_3 e^{-c_4 \sqrt{N}}, \quad (33)$$

for some positive constants  $c_1, c_2, c_3, c_4 > 0$ . Therefore the adaptive bisection-based policy, whose estimation error decreases exponentially in  $N$ , gives a quadratically better exponential rate than the UEP repetition code. This implies that the UEP repetition code gives a MSE exponent (15) and quantized MSE exponent (16) that is equal to zero at any positive rate  $R > 0$  where  $k = NR / \ln 2$  bits.

*Lemma 3: With the UEP repetition code, the best achievable MSE exponent and the quantized MSE exponent are*

$$E_{\text{MSE, repetition}}^*(R) = E_{\text{q, repetition}}^*(R) = 0 \quad (34)$$

*at any positive rate  $R > 0$ .*

For non-adaptive block querying, in order to improve the error rates of convergence we need to use more sophisticated codes that can efficiently encode  $k = O(N)$  information bits in a length  $N$  codeword while guaranteeing reliable transmission of the  $k = O(N)$  bits. For this purpose, we consider a non-adaptive block querying based on random block coding in the following section.

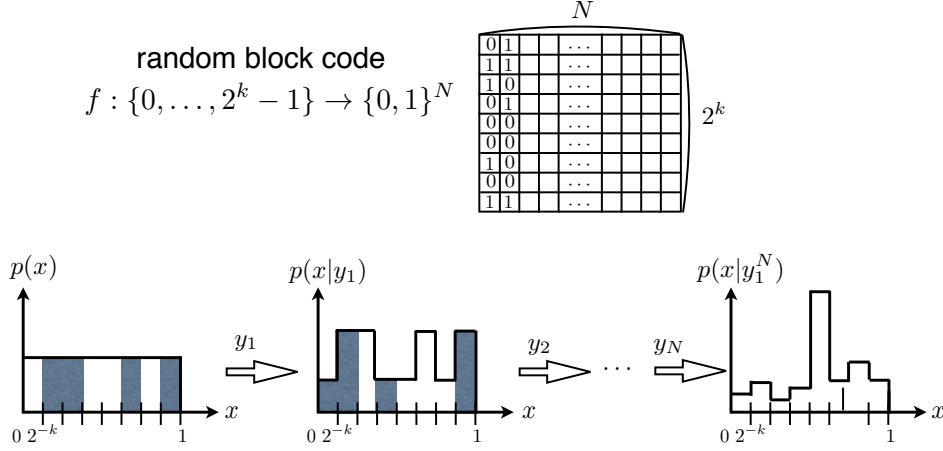


Fig. 4. Non-adaptive block querying based on random block coding with encoder  $f : \{0, \dots, 2^k - 1\} \rightarrow \{0, 1\}^N$ . The  $i$ -th querying region (shade region) is the union of the sub-intervals  $I_{m'} = [m'2^{-k}, (m' + 1)2^{-k})$  for  $m'$  such that the  $i$ -th bit of the corresponding codeword equals 1. Since every symbol of every codeword are i.i.d. with Bernoulli(1/2), at each querying about half of the sub-intervals belong to the querying region. As the querying progresses, if the posterior probability of the event  $\{x \in I_{m'}\}$  for the correct message  $m$  becomes higher than those of the other incorrect messages, the player can correctly decode the index  $m$  of the sub-interval where the value  $x$  of the target variable belongs.

### C. Non-Adaptive Block Querying Based on Random Block Coding

In this section, we introduce a non-adaptive block querying strategy based on random block coding [14]. The encoding map  $f : \{0, \dots, e^{NR} - 1\} \rightarrow \{0, 1\}^N$  of the random block code independently generates length- $N$  codewords  $\mathbf{z}^{(m)} = (z_1^{(m)}, \dots, z_N^{(m)}) := f(m)$  with i.i.d. symbols of Bernoulli(1/2) distribution. The player and the oracle agree on the encoding map, which in turn specifies a block of queries  $(Q_1, \dots, Q_N)$ . Fig. 4 illustrates the one-to-one mapping between the codebook and the block of queries. For a given block code with codewords  $\{\mathbf{z}^{(m)}\}$ ,  $m \in \{0, \dots, 2^k - 1\}$ , the corresponding  $i$ -th querying region  $Q_i$  becomes the union of the intervals  $I_{m'} = [m'2^{-k}, (m' + 1)2^{-k})$  of  $m'$ 's such that the  $i$ -th answer bit  $z_i^{(m')} = 1$ .

When the value of the target variable  $X$  belongs to the sub-interval  $I_m$ , the oracle transmits the length- $N$  answer bits  $\mathbf{z}^{(m)} = (z_1^{(m)}, \dots, z_N^{(m)})$  to the block of queries  $(Q_1, \dots, Q_N)$  by  $N$ -uses of the BSC( $\epsilon$ ). The length- $N$  channel output sequence that the player receives is denoted by  $\mathbf{y} = \mathbf{z}^{(m)} \oplus \mathbf{n}$  where  $\mathbf{n}$  is the noise sequence composed of i.i.d. symbols with Bernoulli( $\epsilon$ ) distribution. Given the channel output sequence  $\mathbf{y}$ , the player decodes the estimate  $\hat{m}$  of  $m$  that maximizes the likelihood (ML decoder)

$$\hat{m} = \arg \max_m p^N(\mathbf{y} | \mathbf{z}^{(m)}) \quad (35)$$

where  $p^N(\mathbf{y} | \mathbf{z}) = \prod_{i=1}^N p_{Y|Z}(y_i | z_i)$  and  $p_{Y|Z}(y | z)$  is the transition probability of the BSC( $\epsilon$ ). Define the set of  $\mathbf{y}$ 's that are mapped to the message  $m'$  by the ML decoder as  $\mathcal{Y}_{m'}$  for  $m' \in \{0, \dots, e^{NR} - 1\}$ . Since the message  $M$



is uniformly distributed over  $\{0, \dots, e^{NR} - 1\}$  for  $X \sim \text{unif}[0, 1]$ , the average decoding error probability is

$$\Pr(\hat{M} \neq M) = \sum_{m=0}^{e^{NR}-1} e^{-NR} \sum_{\mathbf{y} \notin \mathcal{Y}_m} p^N(\mathbf{y}|\mathbf{z}^{(m)}). \quad (36)$$

We review previous results on analyzing the exponentially decreasing rate of  $\Pr(\hat{M} \neq M)$  for random block codes with the ML decoding, and use it to analyze the best achievable quantized MSE exponent (16) with the random block code.

For the random block code of rate  $R$ , define the best achievable error exponent for the block decoding error probability  $\Pr(\hat{M} \neq M)$  as

$$E_r(R) := \liminf_{N \rightarrow \infty} \frac{-\ln \Pr(\hat{M} \neq M)}{N}. \quad (37)$$

For a BSC( $\epsilon$ ) with the optimum input distribution Bernoulli(1/2), Forney's analysis [15] provides a closed form solution for  $E_r(R)$ ,

$$E_r(R) = \begin{cases} E_0(1/2, \epsilon) - R, & 0 \leq R < R_{\text{crit}}(\epsilon), \\ D_B(\gamma_{\text{GV}}(R) \parallel \epsilon), & R_{\text{crit}}(\epsilon) \leq R \leq C, \end{cases} \quad (38)$$

where  $E_0(a, b) = -\log(1 - 2a(1-a)(\sqrt{b} - \sqrt{1-b})^2)$  and thus  $E_0(1/2, \epsilon) = -\log(1/2 + \sqrt{\epsilon(1-\epsilon)})$ ,  $R_{\text{crit}}(\epsilon) = D_B(\gamma_{\text{crit}}(\epsilon) \parallel 1/2)$  with  $\gamma_{\text{crit}}(\epsilon) = \frac{\sqrt{\epsilon}}{\sqrt{\epsilon} + \sqrt{1-\epsilon}}$ ,  $C = H_B(1/2) - H_B(\epsilon)$ , and  $\gamma_{\text{GV}}(R)$  is the normalized Gilbert-Varshamov distance. The exponent  $E_r(R)$  is a decreasing function of the rate  $R$ . As shown in [16] (pp. 147-149), for a very noisy channel ( $\epsilon \approx 0.5$ ) the error exponent in (38) can be approximated as

$$E_r(R) \approx \begin{cases} \frac{C}{2} - R, & 0 \leq R < \frac{C}{4}, \\ (\sqrt{C} - \sqrt{R})^2, & \frac{C}{4} \leq R \leq C. \end{cases} \quad (39)$$

From the upper bound in (19) we obtain the bound on the quantized MSE:

$$\mathbb{E}[c_q(X, \hat{X}_{N, \text{finite}})] \leq e^{-NE_r(R)}. \quad (40)$$

Therefore,  $E_r(R)$  is the achievable quantized MSE exponent. Moreover, we can also show that the exponent  $E_r(R)$  is not only an achievable exponent but also the best achievable quantized MSE exponent with the random block coding. This can be shown by using the lower bound in (24) and the fact that the random block code provides equal protection for every information bit, which makes the exponent of every bit decoding error probability equal to the exponent of the block decoding error, i.e.,

$$\Pr(B_i \neq \hat{B}_i) \doteq \Pr(\hat{M} \neq M), \quad \forall i \in \{1, \dots, k = NR/\ln 2\}. \quad (41)$$

*Lemma 4: The best achievable quantized MSE exponent with the non-adaptive block querying strategy based on random block code is equal to*

$$E_{\text{q,rc}}^*(R) = E_r(R) \quad (42)$$

for the random coding exponent  $E_r(R)$  defined in (38).

*Proof:* Appendix C. ■

Compared to the UEP repetition code that achieves MSE and the quantized MSE decreasing exponentially only in  $\sqrt{N}$ , the block querying based on random block coding achieves the estimation errors exponentially decreasing in  $N$ , which is the same rate achievable by the adaptive bisection policy. However, the random block code is not a MSE-optimal non-adaptive policy since it does not take into account the different contributions of each information bit to the MSE. In the next section, we introduce a new non-adaptive block querying strategy based on superposition coding, which employs both coding gain and unequal error protection.

#### IV. NON-ADAPTIVE BLOCK QUERYING BASED ON SUPERPOSITION CODING

Superposition codes [13] were originally developed as a channel coding scheme for communications over a degraded broadcast channel where one receiver is *stronger* than the other such that the stronger receiver can always recover the weaker receiver's message as well as its own message. The weaker receiver's message is thus treated as a public message and the stronger receiver's message is treated as a private message. Since the public message should be decodable not only to the stronger receiver but also to the weaker receiver, a better error protection is required for the public message than for the private message. We can use the superposition coding scheme to design an unequal error protection code for two levels of priority, where the more important layer contains the public message and the less important layer contains the private message.

In this section, we use this superposition coding principles to develop a non-adaptive block querying strategy that provides better error protection for MSBs than for LSBs and achieves a reliable communication for total  $k = NR/\ln 2$  information bits, which linearly increase in the number of query rounds  $N$  for a fixed rate  $0 < R \leq C$ . By efficiently allocating error protection to the MSBs and the LSBs, the UEP querying strategy achieves better MSE convergence rates as compared to random block codes.

We first partition the information bits  $(B_1, \dots, B_k)$  into two sub-groups, a group containing the first  $k_1 < k$  bits of  $X$   $(B_1, \dots, B_{k_1})$  and the other group containing remaining  $k_2 := k - k_1$  bits of  $X$   $(B_{k_1+1}, \dots, B_{k_1+k_2})$ . The group of MSBs  $(B_1, \dots, B_{k_1})$  determines the more important partial message  $M_1 \in \{0, \dots, 2^{k_1} - 1\}$ , while the group of LSBs  $(B_{k_1+1}, \dots, B_{k_1+k_2})$  determines the less important partial message  $M_2 \in \{0, \dots, 2^{k_2} - 1\}$ . Denote the rates of  $M_1$  and of  $M_2$  by  $R_1 = (k_1 \ln 2)/N$  and  $R_2 = (k_2 \ln 2)/N$ , respectively.

Upon transmission of  $M = (M_1, M_2)$  of total rate  $R = R_1 + R_2$ , the quantized MSE  $\mathbb{E}[c_q(X, \hat{X}_N)]$  can be expressed in terms of the decoding error of the partial messages as

$$\begin{aligned} \mathbb{E}[c_q(X, \hat{X}_N)] &= \Pr(\hat{M}_1 \neq M_1) \mathbb{E}[c_q(X, \hat{X}_N) | \hat{M}_1 \neq M_1] \\ &\quad + \Pr(\hat{M}_1 = M_1, \hat{M}_2 \neq M_2) \mathbb{E}[c_q(X, \hat{X}_N) | \hat{M}_1 = M_1, \hat{M}_2 \neq M_2] \\ &\quad + \Pr(\hat{M}_1 = M_1, \hat{M}_2 = M_2) \mathbb{E}[c_q(X, \hat{X}_N) | \hat{M}_1 = M_1, \hat{M}_2 = M_2]. \end{aligned} \quad (43)$$

When the partial message  $M_1$ , which is composed of the  $NR_1$ -most significant bits of  $X$ , can be correctly decoded, the quantized MSE associated with the error of  $\hat{X}_N = \hat{X}_{N, \text{finite}}$  can be bounded above by  $e^{-2NR_1}$ . By using this bound and the fact that  $\mathbb{E}[c_q(X, \hat{X}_N) | \hat{M}_1 = M_1, \hat{M}_1 = M_2] = 0$ , the quantized MSE can be bounded above as

$$\mathbb{E}[c_q(X, \hat{X}_{N, \text{finite}})] \leq \Pr(\hat{M}_1 \neq M_1) + \Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1) e^{-2NR_1} \quad (44)$$

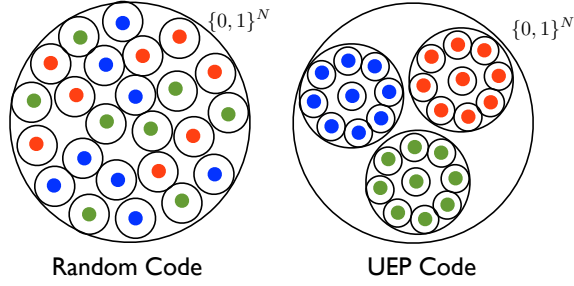


Fig. 5. The distributions of codewords (each color dot) in the output space  $\{0, 1\}^N$  for random block code and for UEP code with two levels of priority. To better protect color information of the codewords, which represents MSBs of the message of the codewords, the same color codewords should be clustered together. However, this clustering makes it harder to decode the correct codeword among the same color codewords, i.e., harder to decode the LSBs of the message.

for  $R_1 < R$ . The weight  $e^{-2NR_1}$  on  $\Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1)$  indicates that the importance of the partial message  $M_2$  conditioned on the correctly decoded  $M_1$  is significantly smaller than that of  $M_1$ .

When we use the random block code, the best achievable decoding error rates for the partial message  $M_1$  (MSBs) and for  $M_2$  (LSBs) conditioned on the correct estimate  $\hat{M}_1 = M_1$  are

$$\begin{aligned} \Pr(\hat{M}_1 \neq M_1) &\doteq e^{-NE_r(R_1+R_2)}, \\ \Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1) &\doteq e^{-NE_r(R_2)} \end{aligned} \quad (45)$$

for the random block code error exponent  $E_r(\cdot)$  in (38). Since  $E_r(R)$  is a decreasing function of the rate  $R$ , the exponents in (45) indicates that  $\Pr(\hat{M}_1 \neq M_1)$  dominates  $\Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1)$  for the random block code. In other words, the decoding error probability  $\Pr(\hat{M}_1 \neq M_1)$  of the partial message  $M_1$  becomes a bottleneck in reducing the upper bound in (44). To improve the quantized MSE exponent compared to that of the random block code, we will design a UEP code whose  $\Pr(\hat{M}_1 \neq M_1)$  decreases at least faster than that of the random block code,  $e^{-NE_r(R_1+R_2)}$ . By using the improved convergence rates of  $\Pr(\hat{M}_1 \neq M_1)$ , we will also demonstrate that the proposed UEP code achieves a gain in the exponentially decreasing rate of the quantized MSE  $\mathbb{E}[c_q(X, \hat{X}_N)]$  for high rate regimes of  $R > 0$ .

#### A. Encoding of Superposition Codes and the Associated Non-Adaptive Block Querying

In Fig. 5, we illustrate the codeword distributions of the random block code and of a desired UEP code with two levels of priority, where the more important layer contains the MSBs and the less important layer contains the LSBs. Each color dot is a codeword, and the shell around it is the decoding region for  $M = (M_1, M_2)$  in the output space  $\{0, 1\}^N$ . Here the partial message  $M_1$  is represented by the color of the codeword. Codewords with the same color have the same partial message  $M_1$  (MSBs), while their  $M_2$ 's (LSBs) are different. For the random block code, the same color codewords are uniformly distributed in  $\{0, 1\}^N$ . When a noise vector corrupts the transmitted codeword beyond the correct decoding region, the decoded message  $\hat{M}$  may not have the same color as that of  $M$ , since there are  $e^{NR_1}$  different colors. On the other hand, if the same color codewords are concentrated together as shown in

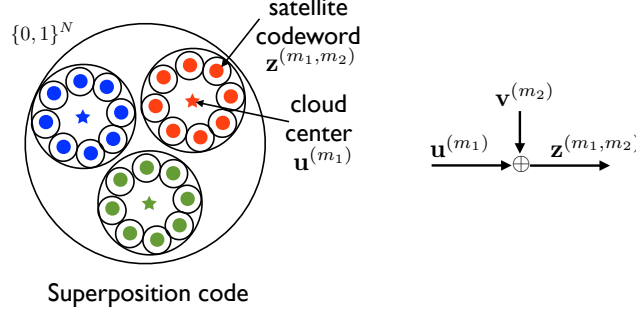


Fig. 6. Superposition coding with two levels of priority, where the first layer encodes MSBs (color information of the codewords) and the second layer encodes LSBs.

the right figure, even if noise corrupts the transmitted codeword, the color information will have higher probability of being correctly decoded. However, the probability of  $M_2$  being correctly decoded given a correct estimate for  $\hat{M}_1 = M_1$  will be lower for the UPE code, since the codewords of the same color are closer to each other. We next construct a code that satisfies such a UEP property for the partial messages  $(M_1, M_2)$  using superposition coding principles.

A superposition code is constructed by superimposing two random block codes generated by different distributions. The first random block code of length  $N$  and rate  $R_1$  is composed of  $e^{NR_1}$  binary length- $N$  codewords,  $\{\mathbf{u}^{(m_1)}\}$ ,  $m_1 \in \{0, \dots, e^{NR_1} - 1\}$ , which encode the more important partial message  $m_1$  (MSBs). The symbols of every codeword are chosen independently at random with Bernoulli(1/2) distribution. We call these partial codewords “cloud centers” in the output space  $\{0, 1\}^N$ . The second random block code of length  $N$  and rate  $R_2$  is composed of codewords  $\{\mathbf{v}^{(m_2)}\}$ ,  $m_2 \in \{0, \dots, e^{NR_2} - 1\}$ , and it encodes the less important partial message  $m_2$  (LSBs). Every symbol of every codeword in  $\{\mathbf{v}^{(m_2)}\}$  is independent and identically distributed with Bernoulli( $\alpha$ ) distribution for some  $\alpha \in (0, 1/2)$ . The codeword  $\mathbf{z}^{(m_1, m_2)}$  for the total message  $(m_1, m_2)$  is designed by the bit-wise XOR of the partial codewords  $\mathbf{u}^{(m_1)}$  and  $\mathbf{v}^{(m_2)}$ . The superposition code  $\mathcal{C}_s$  for the message  $(m_1, m_2) \in \{0, \dots, e^{NR_1} - 1\} \times \{0, \dots, e^{NR_2} - 1\}$  of rate  $R = R_1 + R_2$  is thus composed of  $\{\mathbf{z}^{(m_1, m_2)}\}$ , where  $\mathbf{z}^{(m_1, m_2)} = \mathbf{u}^{(m_1)} \oplus \mathbf{v}^{(m_2)}$ . The codewords  $\{\mathbf{z}^{(m_1, m_2)}\}$  for a fixed  $m_1$  are called “satellite codewords.” There are  $e^{NR_2}$ -satellite codewords around each cloud center  $\mathbf{u}^{(m_1)}$ . Fig. 6 illustrates the construction of superposition code.

Note that when  $\alpha = 1/2$  the distribution of the codewords in  $\mathcal{C}_s$  is the same as that of a random block code that is composed of  $e^{N(R_1+R_2)}$  independent and identically distributed codewords where every symbol of every codeword is chosen independently at random with Bernoulli(1/2) distribution. Therefore, the random block code is a special case of the superposition code. In contrast to the case of  $\alpha = 1/2$ , where every codeword is independent, for a superposition code with  $\alpha \in (0, 1/2)$  the satellite codewords  $\{\mathbf{z}^{(m_1, m_2)}\}$ ,  $m_2 \in \{0, \dots, e^{NR_2} - 1\}$ , for a fixed  $m_1$  (the same color codewords), are mutually dependent. Since the typical Hamming weight of  $\mathbf{v}^{(m_2)}$  is  $N\alpha$ , the typical distance between a satellite codeword  $\mathbf{z}^{(m_1, m_2)} = \mathbf{u}^{(m_1)} \oplus \mathbf{v}^{(m_2)}$  and its cloud center  $\mathbf{u}^{(m_1)}$  is  $N\alpha$ . As  $\alpha$  decreases from 1/2 to 0, the satellite codewords become more and more concentrated around its cloud center.

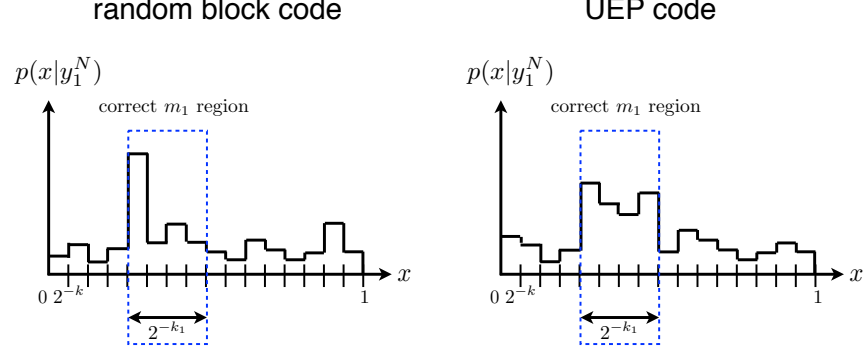


Fig. 7. Illustration of the typical posterior distributions of  $X$  after  $N$  rounds of querying for random block coding (left) and for UEP coding with two levels of priority (right). Consider the case where the posterior probability of  $X$  in the correct  $m_1$  region is higher for the UEP coding compared to that of the random block coding but the peaks within the correct  $m_1$  region are smooth for the UEP coding compared to those of the random block code. For such a case, the UEP coding provides better error protection for  $m_1$  but worse error protection for  $m_2$  given the correct  $\hat{m}_1 = m_1$  than does the random block code.

Therefore, the superposition code satisfies the desired geometric property of UEP with two levels of priority. The parameter  $\alpha \in (0, 1/2)$  determines how much the satellite codewords are concentrated around its cloud center, which determines the trade-offs between decoding error probability of  $M_1$  and that of  $M_2$ .

The block of querying regions  $(Q_1, \dots, Q_N)$  associated with the superposition codewords  $\{\mathbf{z}^{(m_1, m_2)} = \mathbf{u}^{(m_1)} \oplus \mathbf{v}^{(m_2)}\}$ ,  $m_1 \in \{0, \dots, 2^{k_1} - 1\}$ ,  $m_2 \in \{0, \dots, 2^{k_2} - 1\}$ , can be represented in terms of the sub-intervals  $I_{m_1} := \{m_1 2^{-k_1}, (m_1 + 1) 2^{-k_1}\}$  of resolution  $2^{-k_1}$  and  $I_{m_1, m_2} := \{m_1 2^{-k_1} + m_2 2^{-k}, m_1 2^{-k_1} + (m_2 + 1) 2^{-k}\}$  of resolution  $2^{-k}$  as

$$\begin{aligned}
 Q_i &= \bigcup_{\{(m_1, m_2): z_i^{(m_1, m_2)} = 1\}} I_{m_1, m_2} \\
 &= \left( \bigcup_{\{m_1: u_i^{(m_1)} = 1\}} \left( I_{m_1} \cap \left( \bigcup_{\{m_2: v_i^{(m_2)} = 0\}} I_{m_1, m_2} \right) \right) \right) \cup \left( \bigcup_{\{m_1: u_i^{(m_1)} = 0\}} \left( I_{m_1} \cap \left( \bigcup_{\{m_2: v_i^{(m_2)} = 1\}} I_{m_1, m_2} \right) \right) \right).
 \end{aligned} \tag{46}$$

Since  $v_i^{(m_2)}$  is i.i.d. with Bernoulli( $\alpha$ ),  $\alpha \in (0, 1/2)$ , for a fixed  $m_1$  such that  $u_i^{(m_1)} = 1$  about a  $(1 - \alpha)$ -fraction of the sub-intervals  $I_{m_1, m_2}$ ,  $m_2 \in \{0, \dots, 2^{k_2} - 1\}$ , within the  $I_{m_1}$  is included in  $Q_i$ . On the other hand, if  $u_i^{(m_1)} = 0$  for some  $m_1$ , about a  $\alpha$ -fraction of the sub-intervals  $I_{m_1, m_2}$ ,  $m_2 \in \{0, \dots, 2^{k_2} - 1\}$ , within the  $I_{m_1}$  is included in  $Q_i$ . Therefore, different from block querying based on random block coding, where each of the sub-intervals  $I_{m_1, m_2}$ ,  $m_1 \in \{0, \dots, 2^{k_1} - 1\}$ ,  $m_2 \in \{0, \dots, 2^{k_2} - 1\}$  is independently either included in or excluded from  $Q_i$  with probabilities  $1/2$  and  $1/2$ , for block querying based on superposition coding the events  $\{I_{m_1, m_2} \subset Q_i\}$ ,  $m_2 \in \{0, \dots, 2^{k_1} - 1\}$ , for a fixed  $m_1$ , depend on each other.

Fig. 7 illustrates how the posterior distribution of  $X$  after  $N$  rounds of querying with the UEP coding (right figure) will appear as compared to that of the random block coding (left figure). Given the channel outputs  $y_1^N$ , if the posterior probability taken by  $m'_1$  region, i.e.,  $X \in [m'_1 2^{-k_1}, (m'_1 + 1) 2^{-k_1})$ , is largest for the correct  $m'_1 = m_1$

among all the  $m'_1 \in \{0, \dots, 2^{k_1} - 1\}$ , then the partial message  $m_1$  can be correctly decoded by optimal ML decoder for the partial message  $m_1$ . If the posterior probability associated with the correct  $m_1$  region is higher for UEP coding than it is for random block coding, one obtains an improvement in the decoding error probability  $\Pr(\hat{M}_1 \neq M_1)$  of the partial message  $M_1$ . However, this improvement might come at the cost of degraded  $\Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1)$ .

### B. Decoding of Superposition Code and Error Exponents of Partial Messages

In this section, we show that the non-adaptive block querying based on the superposition code can achieve an improved error exponent for the decoding error probability  $\Pr(\hat{M}_1 \neq M_1)$  of the more important partial message  $M_1$  (MSBs) as compared to that of the random block code. This improvement occurs for sufficiently small  $R_1$  and sufficiently large  $R_2 < C_2(\alpha) := H_B(\alpha * \epsilon) - H_B(\epsilon)$ .

Denote the maximum achievable error exponents of  $\Pr(\hat{M}_1 \neq M_1)$  and of  $\Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1)$  with the superposition code of rates  $(R_1, R_2)$  as

$$\begin{aligned} E_{\text{MSBs}}^*(R_1, R_2, \alpha) &= \liminf_{N \rightarrow \infty} \frac{-\ln \Pr(\hat{M}_1 \neq M_1)}{N}, \\ E_{\text{LSBs}}^*(R_2, \alpha) &= \liminf_{N \rightarrow \infty} \frac{-\ln \Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1)}{N}, \end{aligned} \quad (47)$$

where  $\alpha \in (0, 1/2)$  is the parameter that determines the distribution of the codewords as explained in Section IV-A. We analyze these exponents and compare those to the best achievable decoding error exponents (45) of the random block code.

There have been many previous works [17], [18], [19] to analyze the error exponents  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$  and  $E_{\text{LSBs}}^*(R_2, \alpha)$  of the superposition code. A lower bound on  $E_{\text{LSBs}}^*(R_2, \alpha)$  can be calculated by directly applying the Gallager's error exponent analysis for a discrete memoryless channel with random block code of i.i.d. symbols having Bernoulli( $\alpha$ ) distribution [19]. On the other hand, the analysis of  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$  is much more complicated, since it involves comparisons between sum of likelihoods of exponentially many codewords (the satellite codewords in Fig. 6), which are dependent on each other, in order to find the most probable  $M_1$  (MSBs, or the color of the transmitted codeword). Even though there exist a few lower bounds on  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$  and some bounds are shown to be numerically tighter than the others, there has been no simple closed form solution for error exponents for superposition code. Since our goal is not to exactly calculate the error exponent  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$  of  $\Pr(\hat{M}_1 \neq M_1)$  but to prove a gain in the quantized MSE exponent from the UEP superposition coding, we consider two well-known sub-optimal decoding rules instead of the ML decoding rule. This will provide lower bounds on the error exponents for ML decoding of UEP superposition codes for the noisy 20 questions problem.

The first sub-optimal decoding rule we consider is joint maximum likelihood (JML) decoding for  $m = (m_1, m_2)$ . Given the received word  $\mathbf{y} = \mathbf{z}^{(m_1, m_2)} \oplus \mathbf{n}$ , which is the noisy version of the block of answers  $\mathbf{z}^{(m_1, m_2)}$  added by a length- $N$  noise  $\mathbf{n}$  with i.i.d. Bernoulli( $\epsilon$ ) symbols, this decoding rule finds the most probable  $(\hat{m}_1, \hat{m}_2)$  such that

$$(\hat{m}_1, \hat{m}_2) = \arg \max_{(m_1, m_2)} p^N(\mathbf{y} | \mathbf{z}^{(m_1, m_2)}) \quad (48)$$

where  $p^N(\mathbf{y}|\mathbf{z}) = \prod_{i=1}^N p_{Y|Z}(y_i|z_i)$  and  $p_{Y|Z}(y|z)$  is the transition probability of the BSC( $\epsilon$ ). Note that this decoding rule minimizes the probability of block decoding error  $(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)$  but not the probability of the partial decoding error  $\hat{M}_1 \neq M_1$ , so that this is a sub-optimal decoding rule for  $M_1$ . The decoding error of  $M_1$  happens only when  $\hat{M}_1 \neq M_1$ , regardless of whether  $\hat{M}_2 \neq M_2$  or not. Let  $E_{\text{MSBs,JML}}(R_1, R_2, \alpha)$  denote the best achievable error exponent of  $\Pr(\hat{M}_1 \neq M_1)$  with JML decoding. In Lemma 5, we show that  $E_{\text{MSBs,JML}}(R_1, R_2, \alpha) \geq E_r(R_1 + R_2)$  for every  $(R_1, R_2)$ , regardless of the choice of  $\alpha \in (0, 1/2)$ . This implies that the superposition code provides a better, or at least as good, error protection for the partial message  $M_1$  as that of the random block code for every  $(R_1, R_2)$ , regardless of the choice of  $\alpha \in (0, 1/2)$ .

The second sub-optimal decoding rule we consider is successive cancellation (SC) decoding. To decode  $\hat{m}_1$ , this decoding rule focuses only on the geometry of the partial codewords  $\{\mathbf{u}^{(m_1)}\}$ ,  $m_1 \in \{0, \dots, e^{NR_1} - 1\}$ , (cloud centers in Fig. 6) while ignoring the true structure of the overall codewords  $\{\mathbf{z}^{(m_1, m_2)}\}$ . More specifically, this decoding rule behaves as if one of  $\{\mathbf{u}^{(m_1)}\}$  is transmitted and the received word  $\mathbf{y}$  is corrupted by a noise word  $\mathbf{v}^{(m_2)} \oplus \mathbf{n}$ . Note that  $\mathbf{u}^{(m_1)}$ ,  $\mathbf{v}^{(m_2)}$ , and  $\mathbf{n}$  are independent of each other, and every symbol of  $\mathbf{v}^{(m_2)}$  are i.i.d. Bernoulli( $\alpha$ ) and every symbol of every  $\mathbf{n}$  are i.i.d. Bernoulli( $\epsilon$ ). Therefore, the new noise word  $\mathbf{v}^{(m_2)} \oplus \mathbf{n}$  can be modeled as a sequence of i.i.d. symbols following a Bernoulli( $\alpha * \epsilon$ ) distribution where  $\alpha * \epsilon = \alpha(1 - \epsilon) + (1 - \alpha)\epsilon$ . Denoting by  $q_{Y|U}(y|u)$  the transition probability of the BSC( $\alpha * \epsilon$ ) and defining  $q^N(\mathbf{y}|\mathbf{u}) = \prod_{i=1}^n q_{Y|U}(y_i|u_i)$ , this sub-optimal decoding rule produces the estimate  $\hat{m}_1$  of the MSBs in the dyadic expansion of  $\mathbf{X}$ :

$$\hat{m}_1 = \arg \max_{m_1} q^N(\mathbf{y}|\mathbf{u}^{(m_1)}) \quad (49)$$

for a given channel output sequence  $\mathbf{y}$ . After decoding  $m_1$  and having the estimate  $\hat{m}_1$ , the SC decoding rule subtracts  $\mathbf{u}^{(\hat{m}_1)}$  from  $\mathbf{y}$  and finds the estimate  $\hat{m}_2$  for the partial message  $m_2$  (LSBs) that maximizes the likelihood of  $p^N(\mathbf{y} \oplus \mathbf{u}^{(\hat{m}_1)}|\mathbf{v}^{(m_2)})$

$$\hat{m}_2 = \arg \max_{m_2} p^N(\mathbf{y} \oplus \mathbf{u}^{(\hat{m}_1)}|\mathbf{v}^{(m_2)}). \quad (50)$$

Let  $E_{\text{MSBs,SC}}(R_1, \alpha)$  and  $E_{\text{LSBs,SC}}(R_2, \alpha)$  denote the best achievable error exponents of  $\Pr(\hat{M}_1 \neq M_1)$  and of  $\Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1)$ , respectively, with SC decoding. Forney's analysis [15] yields the exponentially tight error exponent  $E_{\text{MSBs,SC}}(R_1, R_2, \alpha)$  for  $\Pr(\hat{M}_1 \neq M_1)$  with the SC decoding rule

$$E_{\text{MSBs,SC}}(R_1, \alpha) = \begin{cases} E_0(1/2, \alpha * \epsilon) - R_1, & 0 \leq R_1 \leq R_{\text{crit}}(\alpha * \epsilon), \\ D_B(\gamma_{\text{GV}}(R_1) \|\alpha * \epsilon), & R_{\text{crit}}(\alpha * \epsilon) < R_1 \leq C - C_2(\alpha). \end{cases} \quad (51)$$

Here  $E_0(a, b) = -\ln(1 - 2a(1 - a)(\sqrt{b} - \sqrt{1 - b})^2)$  and thus  $E_0(1/2, \alpha * \epsilon) = -\ln(1/2 + \sqrt{(\alpha * \epsilon)(1 - (\alpha * \epsilon))})$ ,  $C = H_B(1/2) - H_B(\epsilon)$ ,  $C_2(\alpha) = H_B(\alpha * \epsilon) - H_B(\epsilon)$ ,  $R_{\text{crit}}(\alpha * \epsilon) = D_B(\gamma_{\text{crit}}(\alpha * \epsilon) \| 1/2)$  and  $\gamma_{\text{crit}}(\alpha * \epsilon) = \frac{\sqrt{\alpha * \epsilon}}{\sqrt{\alpha * \epsilon} + \sqrt{1 - \alpha * \epsilon}}$ . Moreover, for a given  $\alpha \in (0, 1/2)$ , the error exponent  $E_{\text{LSBs,SC}}(R_2, \alpha)$  of the partial message  $M_2$  can be shown to be positive for every  $0 \leq R_2 < C_2(\alpha)$ .

The following lemma summarizes the lower bounds on the error exponent  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$  (47) achievable with sub-optimal JML decoding and with sub-optimal SC decoding.

*Lemma 5: The superposition code with ML decoding provides a better, or at least as good, error protection for the partial message  $M_1$  as that of the random block code for every  $(R_1, R_2)$ , regardless of the choice of  $\alpha \in (0, 1/2)$ , i.e.,*

$$E_{\text{MSBs}}^*(R_1, R_2, \alpha) \geq E_{\text{MSBs,JML}}(R_1, R_2) \geq E_r(R_1 + R_2) \quad (52)$$

where  $E_{\text{MSBs,JML}}(R_1, R_2)$  is the best achievable exponent using JML decoding rule. For sufficiently small  $R_1 > 0$  and sufficiently large  $R_2 < C_2(\alpha) = H_B(\alpha * \epsilon) - H_B(\epsilon)$ , a strictly positive gain in the exponent can be achieved using SC decoding rules, i.e.,

$$E_{\text{MSBs}}^*(R_1, R_2, \alpha) \geq E_{\text{MSBs,SC}}(R_1, \alpha) > E_r(R_1 + R_2) \quad (53)$$

where  $E_{\text{MSBs,SC}}(R_1, \alpha)$  is the best achievable exponent using SC decoding rule.

*Proof:* Appendix D. ■

For a very noisy BSC( $\epsilon$ ) with capacity  $C = H_B(1/2) - H_B(\epsilon)$ , we can further demonstrate that, when we choose  $R_2 = C_2(\alpha)$ , for the entire regime of  $R_1 \in [0, C - C_2(\alpha)]$  where  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$  is positive, UEP superposition coding provides a strictly positive gain in the error exponent of  $\Pr(\hat{M}_1 \neq M_1)$  as compared to that of the random block code.

*Lemma 6: For a very noisy BSC( $\epsilon$ ) with  $\epsilon = 0.5 - \delta$  for small  $\delta > 0$ , assume a fixed  $\alpha \in (0, 1/2)$  and the rate  $R_2 = C_2(\alpha)$ . Then the best achievable error exponent of  $\Pr(\hat{M}_1 \neq M_1)$  for the superposition code, denoted  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$ , is strictly larger than that of the random block code for every  $R_1 \in [0, C - C_2(\alpha)]$ . More precisely, for every  $R_1 \in [0, C - C_2(\alpha)]$ , with the SC decoding rule we can achieve an error exponent  $E_{\text{MSBs,SC}}(R_1, \alpha)$  strictly larger than  $E_r(R_1 + R_2)$ ,*

$$E_{\text{MSBs}}^*(R_1, R_2, \alpha) \geq E_{\text{MSBs,SC}}(R_1, \alpha) > E_r(R_1 + R_2). \quad (54)$$

*Proof:* Appendix E. ■

In Fig 8, we provide a plot of  $E_r(R_1 + R_2)$  (black solid line) and  $E_{\text{MSBs,SC}}(R_1, \alpha)$  (blue dash-dot line) over  $R = R_1 + R_2$  for a BSC(0.45) for  $\alpha = 0.11$  and  $R_2 = C_2(\alpha)$ . The plot for  $E_{\text{MSBs,SC}}(R_1, \alpha)$  starts from  $R = C_2(\alpha)$  at which  $R_1 = 0$ . It can be shown that  $E_{\text{MSBs,SC}}(R_1, \alpha)$  is above  $E_r(R_1 + R_2)$  for every  $R_1 \in [0, C - C_2(\alpha)]$ .

### C. Gain in the Achievable Quantized MSE Exponent and the MSE Exponent from Superposition coding

By using the improvement in the decoding error exponents of  $M_1$  (MSBs) from superposition coding, we can now demonstrate a gain in the exponentially decreasing rate of the quantized MSE  $\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})]$  in  $N$  for resolution of  $k = NR / \ln 2$  bits. Define  $E_{\text{q,spc}}^*(R)$  the best achievable exponentially decreasing rate of  $\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})]$  with non-adaptive block querying based on the superposition code (SPC) of rate  $R$ :

$$E_{\text{q,spc}}^*(R) = \liminf_{N \rightarrow \infty} \frac{-\ln \mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})]}{N}. \quad (55)$$

The random block code achieves quantized MSE exponent  $E_{\text{q,rc}}^*(R)$  equal to  $E_r(R)$  as demonstrated in Lemma 4. By using Lemma 6, we next show that for a very noisy BSC at high rate regimes of  $R \in (\frac{C}{6}, C)$ , we can achieve  $E_{\text{q,spc}}^*(R) > E_{\text{q,rc}}^*(R) = E_r(R)$  with non-adaptive block querying based on the superposition code.



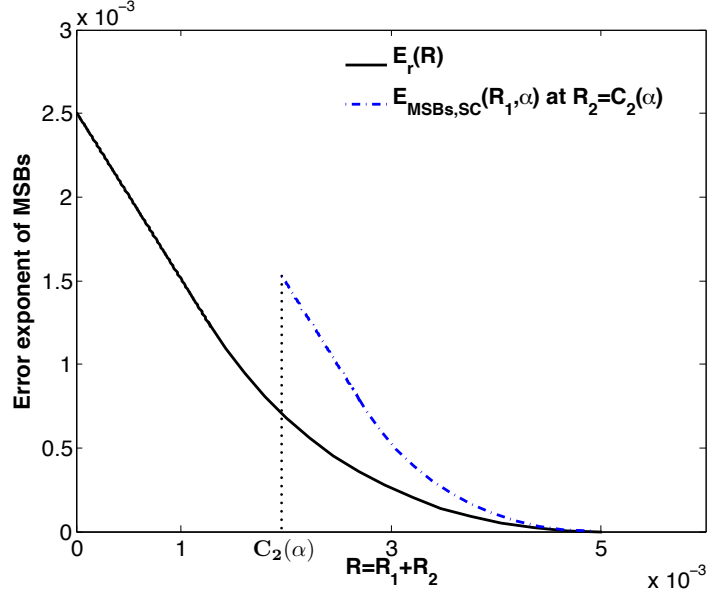


Fig. 8. A plot of  $E_r(R_1 + R_2)$  and  $E_{\text{MSBs,SC}}(R_1, \alpha)$  at  $R_2 = C_2(\alpha)$  where  $\epsilon = 0.45$  and  $\alpha = 0.11$ . When  $R_2 = C_2(\alpha)$ , for every  $R_1 \in [0, C - C_2(\alpha))$ ,  $E_{\text{MSBs,SC}}(R_1)$  is above  $E_r(R_1 + R_2)$ .

*Theorem 1:* For a very noisy BSC( $\epsilon$ ) with  $\epsilon = 0.5 - \delta$  for small  $\delta > 0$ , the quantized MSE exponent  $E_{\text{q,spc}}^*(R)$  for the superposition code is strictly larger than that of the random block code  $E_{\text{q,rc}}^*(R)$  for every  $R \in (\frac{C}{6}, C)$ . More precisely, for any  $R \in (\frac{C}{6}, C)$  there exists a lower bound on  $E_{\text{q,spc}}^*(R)$ , denoted  $E_{\text{q,spc}}(R)$ , that is strictly larger than the exponent  $E_{\text{q,rc}}^*(R)$  of the random block code, i.e.,

$$E_{\text{q,spc}}^*(R) \geq E_{\text{q,spc}}(R) > E_{\text{q,rc}}^*(R) = E_r(R). \quad (56)$$

Here the exponent  $E_{\text{q,spc}}(R)$  is equal to  $E_{\text{MSBs,SC}}(R_1^*(\alpha^*), \alpha^*)$  in (51) where  $\alpha^*$  satisfies  $C_2(\alpha^*) = \frac{6}{5}(R - \frac{C}{6})$  and  $R_1^*(\cdot)$  is defined as  $R_1^*(\alpha) := R - C_2(\alpha)$  for  $C_2(\alpha) = H_B(\alpha * \epsilon) - H_B(\epsilon)$ . This exponent is achievable with the SC decoding rule.

*Proof:* The quantized MSE can be bounded above in terms of block decoding error probabilities of  $M_1$  (MSBs) and  $M_2$  (LSBs) as

$$\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})] \leq \Pr(\hat{M}_1 \neq M_1) + \Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1) e^{-2NR_1}. \quad (57)$$

In Lemma 6, we showed that the decoding error probability of the partial message  $M_1$  at the superposition code rates  $(R_1, R_2)$  with the distribution parameter  $\alpha \in (0, 1/2)$  can be bounded above as

$$\Pr(\hat{M}_1 \neq M_1) \leq e^{-NE_{\text{MSBs,SC}}(R_1, \alpha)} \quad (58)$$

where

$$E_{\text{MSBs,SC}}(R_1, \alpha) = \begin{cases} E_0(1/2, \alpha * \epsilon) - R_1, & 0 \leq R_1 \leq R_{\text{crit}}(\alpha * \epsilon), \\ D_B(\gamma_{\text{GV}}(R_1) \| \alpha * \epsilon), & R_{\text{crit}}(\alpha * \epsilon) < R_1 \leq C - C_2(\alpha). \end{cases} \quad (59)$$

By using this bound and the bound on the conditional decoding error probability of  $M_2$ ,

$$\Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1) \leq e^{-N E_{\text{LSBs,SC}}(R_2, \alpha)}, \quad (60)$$

the quantized MSE with the superposition coding can be bounded above as

$$\mathbb{E}[c_q(X, \hat{X}_{N, \text{finite}})] \leq e^{-N E_{\text{MSBs,SC}}(R_1, \alpha)} + e^{-N E_{\text{LSBs,SC}}(R_2, \alpha)} e^{-2N R_1} \doteq e^{-N \min\{E_{\text{MSBs,SC}}(R_1, \alpha), E_{\text{LSBs,SC}}(R_2, \alpha) + 2R_1\}}.$$

Therefore, the best achievable quantized MSE exponent  $E_{\text{q,spc}}^*(R)$  with the superposition code of rate  $R = R_1 + R_2$  is bounded below as

$$E_{\text{q,spc}}^*(R) \geq \max_{(R_1, R_2, \alpha) \in B} \min\{E_{\text{MSBs,SC}}(R_1, \alpha), E_{\text{LSBs,SC}}(R_2, \alpha) + 2R_1\} \quad (61)$$

where  $B := \{(R_1, R_2, \alpha) : R_1 + R_2 = R, \alpha \in (0, 1/2)\}$ . When we choose  $(R_1, R_2, \alpha) = (R - C_2(\alpha), C_2(\alpha), \alpha)$  for a fixed  $\alpha \in (0, 1/2)$ , since  $E_{\text{LSBs,SC}}(R_2, \alpha) = 0$  at  $R_2 = C_2(\alpha)$ , the right hand side of the lower bound (61) becomes

$$\min\{E_{\text{MSBs,SC}}(R_1, \alpha), E_{\text{LSBs,SC}}(R_2, \alpha) + 2R_1\} = \min\{E_{\text{MSBs,SC}}(R_1^*(\alpha), \alpha), 2R_1^*(\alpha)\} \quad (62)$$

where  $R_1^*(\alpha) := R - C_2(\alpha)$ . Therefore, for this choice of the parameters,  $E_{\text{q,spc}}^*(R)$  can be bounded below as

$$E_{\text{q,spc}}^*(R) \geq \max_{\alpha \in (0, 1/2)} \min\{E_{\text{MSBs,SC}}(R_1^*(\alpha), \alpha), 2R_1^*(\alpha)\}. \quad (63)$$

We next find  $\alpha \in (0, 1/2)$  that maximizes  $\min\{E_{\text{MSBs,SC}}(R_1^*(\alpha), \alpha), 2R_1^*(\alpha)\}$ . More specifically, we find  $\alpha$  that makes  $E_{\text{MSBs,SC}}(R_1^*(\alpha), \alpha) = 2R_1^*(\alpha)$  for  $R_1^*(\alpha) = R - C_2(\alpha)$ , i.e.,  $\alpha$  that satisfies

$$E_{\text{MSBs,SC}}(R - C_2(\alpha), \alpha) = 2(R - C_2(\alpha)). \quad (64)$$

For a very noisy BSC( $\epsilon$ ) with  $\epsilon \approx 1/2$ , as shown in [16] (pp. 147-149), the error exponent  $E_{\text{MSBs,SC}}(R_1, \alpha)$  in (51) can be approximated as

$$E_{\text{MSBs,SC}}(R_1, \alpha) \approx \begin{cases} \frac{C - C_2(\alpha)}{2} - R_1, & 0 \leq R_1 < \frac{C - C_2(\alpha)}{4}, \\ (\sqrt{C - C_2(\alpha)} - \sqrt{R_1})^2, & \frac{C - C_2(\alpha)}{4} \leq R_1 \leq C - C_2(\alpha). \end{cases} \quad (65)$$

By using this approximation, (64) can be written as

$$\frac{C - C_2(\alpha)}{2} - (R - C_2(\alpha)) = 2(R - C_2(\alpha)), \quad (66)$$

which is equivalent to

$$C_2(\alpha) = \frac{6}{5} \left( R - \frac{C}{6} \right). \quad (67)$$

For  $R \in (\frac{C}{6}, C)$ , there always exists a unique  $\alpha \in (0, 1/2)$  satisfying (67). Therefore, with  $\alpha = \alpha^*$  satisfying (67) we have

$$E_{\text{q,spc}}^*(R) \geq \max_{\alpha \in (0, 1/2)} \min\{E_{\text{MSBs,SC}}(R_1^*(\alpha), \alpha), 2R_1^*(\alpha)\} = E_{\text{MSBs,SC}}(R_1^*(\alpha^*), \alpha^*). \quad (68)$$

Lastly, as shown in Lemma 6, since  $E_{\text{MSBs,SC}}(R_1, \alpha) > E_{\text{q,rc}}^*(R) = E_r(R_1 + R_2)$  for every  $R_1 \in [0, C - C_2(\alpha)]$  when  $R_2 = C_2(\alpha)$ , for our choice of  $R_1^*(\alpha^*) = R - C_2(\alpha^*)$  and  $R_2 = C_2(\alpha^*)$ ,

$$E_{\text{MSBs,SC}}(R_1^*(\alpha^*), \alpha^*) > E_r(R). \quad (69)$$

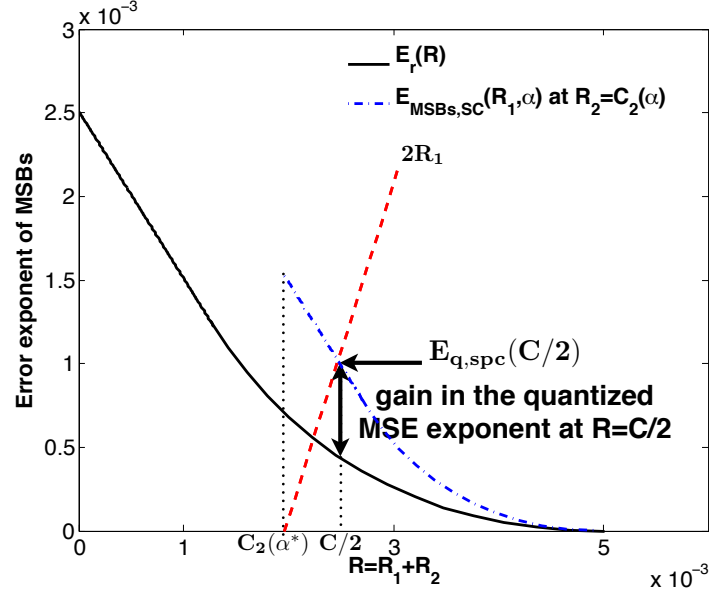


Fig. 9. A plot of  $E_r(R_1 + R_2)$ ,  $E_{\text{MSBs,SC}}(R_1, \alpha)$ , and  $2R_1$  at  $R_2 = C_2(\alpha)$  where  $\epsilon = 0.45$  and  $\alpha = 0.11$ . For  $R = C/2 = 0.0025$ , when we choose  $\alpha$  that makes  $E_{\text{MSBs,SC}}(R_1, \alpha) = 2R_1 = 2(R - C_2(\alpha))$ , the achievable quantized MSE exponent with the superposition code  $E_{\text{q,spc}}(R)$  equals the value at the crossing point of  $E_{\text{MSBs,SC}}(R_1, \alpha)$  and  $2R_1$ , which is strictly larger than the best achievable quantized MSE exponent with the random block code,  $E_r(R)$ .

This completes the proof of the theorem. ■

In Fig 9, the gain in the quantized MSE exponent from superposition code at  $R = C/2$  is illustrated. When we choose  $\alpha$  that makes  $E_{\text{MSBs,SC}}(R_1, \alpha) = 2R_1 = 2(R - C_2(\alpha))$  at  $R = C/2$ , the achievable quantized MSE exponent with the superposition code, which is denoted  $E_{\text{q,spc}}(C/2)$ , equals the value of  $E_{\text{MSBs,SC}}(R_1, \alpha)$  at  $R_1$  where  $E_{\text{MSBs,SC}}(R_1, \alpha)$  and  $2R_1$  cross each other. This value is strictly larger than the best achievable quantized MSE exponent with the random block code,  $E_r(C/2)$ .

We next consider the achievable MSE  $\mathbb{E}[|X - \hat{X}_{N,\text{finite}}|^2]$  with the superposition coding and demonstrate a gain in the MSE exponent in the high rate regimes of the rate  $R$  as compared to that of the random block coding. As shown in (12) and (17) the MSE can be written as a sum of the quantized MSE  $\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})] \doteq e^{-N E_{\text{q,policy}}^*(R)}$  and the finite resolution quantization error  $e^{-2NR}$  as

$$\mathbb{E}[|X - \hat{X}_{N,\text{finite}}|^2] \doteq e^{-N E_{\text{q,policy}}^*(R)} + e^{-N 2R} \doteq e^{-N \min\{E_{\text{q,policy}}^*(R), 2R\}}. \quad (70)$$

As shown in (17), the MSE exponent  $E_{\text{MSE,policy}}^*(R)$  and the quantized MSE exponent  $E_{\text{q,policy}}^*(R)$  at a fixed rate  $R > 0$  are related as

$$E_{\text{MSE,policy}}^*(R) = \min\{E_{\text{q,policy}}^*(R), 2R\}. \quad (71)$$

When  $E_{\text{q,policy}}^*(R) > 2R$ , the MSE exponent at a fixed rate  $R$  is limited by the quantization error from the finite resolution of  $R$  bits, whereas when  $E_{\text{q,policy}}^*(R) \leq 2R$  the MSE exponent is governed by the quantized MSE

TABLE I  
COMPARISON OF FOUR DIFFERENT QUERYING POLICIES

Policy	MSE convergence rate	Features
Bisection policy	$e^{-c_1 N}$ , $c_1 > 0$	Adaptive
Repetition policy	$e^{-c_2 \sqrt{N}}$ , $c_2 > 0$	Non-adaptive, unequal error protection, no coding gain
Random block coding	$e^{-c_3 N}$ , $c_3 > 0$	Non-adaptive, equal error protection, coding gain
Superposition coding	$e^{-c_4 N}$ , $c_4 > 0$	Non-adaptive, unequal error protection, coding gain

exponent, which depends on decoding error.

For random block coding, since the quantized MSE exponent  $E_{q,rc}^*(R)$  is equal to  $E_r(R)$ , the MSE exponent  $E_{MSE,rc}^*(R)$  is

$$E_{MSE,rc}^*(R) = \min\{E_r(R), 2R\}. \quad (72)$$

For a very noisy BSC( $\epsilon$ ), the random block coding error exponent  $E_r(R)$  can be approximated as in (38). By using this approximation, we can show that, where  $\epsilon \in [0.5 - \delta, 0.5]$  for small  $\delta > 0$

$$E_{MSE,rc}^*(R) = \begin{cases} 2R, & 0 \leq R \leq \frac{C}{6}, \\ E_r(R), & \frac{C}{6} < R \leq C. \end{cases} \quad (73)$$

By using Theorem 1, which shows that  $E_{q,spc}(R) > E_{q,rc}^*(R) = E_r(R)$  for  $R \in (C/6, C)$ , we can also show that the MSE exponent  $E_{MSE,spc}^*(R)$  with superposition coding is strictly larger than the MSE exponent  $E_{MSE,rc}^*(R)$  of random block coding in  $R \in (C/6, C)$ .

*Corollary 1: For a very noisy BSC( $\epsilon$ ) with  $\epsilon = 0.5 - \delta$  for small  $\delta > 0$ , the MSE exponent  $E_{MSE,spc}^*(R)$  with the superposition coding is strictly larger than that of the random block code  $E_{MSE,rc}^*(R)$  for every  $R \in (\frac{C}{6}, C)$ , i.e.,*

$$E_{MSE,spc}^*(R) > E_{MSE,rc}^*(R) = E_r(R), \quad \frac{C}{6} < R < C. \quad (74)$$

Therefore, the superposition coding gain in the quantized MSE exponent in the high rate regime  $R \in (C/6, C)$  also results in a gain in the MSE exponent in this regime.

## V. COMPARISONS BETWEEN PERFORMANCE OF THE FOUR DIFFERENT QUERYING POLICIES

In this section, we summarize and compare the four different querying policies discussed in this paper, adaptive bisection policy (Section III-A), non-adaptive repetition policy (Section III-B), non-adaptive block querying based on random block coding (Section III-C), and non-adaptive block querying based on superposition coding (Section IV). Table I summarizes the MSE convergence rate and features of the four policies. Only the bisection policy uses past answers from the oracle to design the next query, while the other three policies determine a set of queries non-adaptively. Among the three non-adaptive block querying policies, repetition policy and the policy based on superposition coding provide unequal error protection for MSBs vs. LSBs of the binary expansion of the target

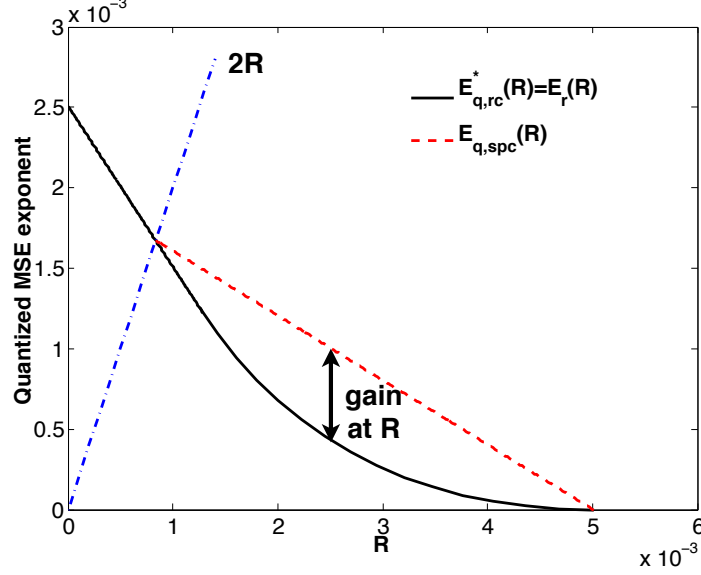


Fig. 10. A plot of  $E_r(R)$ ,  $E_{q,spc}(R)$ , and  $2R$  for BSC( $\epsilon$ ) with  $\epsilon = 0.45$  where  $E_r(R)$  is the best achievable quantized MSE exponent with random block coding and  $E_{q,spc}(R)$  is the lower bound on the best achievable quantized MSE exponent  $E_{q,spc}^*(R)$  with superposition coding. For  $R \in (C/6/C)$ , there exists a gain in the achievable error exponent from the superposition coding.

variable, while the block querying policy based on random block coding provides equal error protection for every information bit. Repetition policy achieves MSE exponentially decreasing only in  $\sqrt{N}$ , while the other two non-adaptive block querying policies as well as the bisection policy achieve the linear in  $N$  exponential rate of decrease. This is because the repetition policy can extract only  $k = O(\sqrt{N})$  information bits reliably by  $N$  number of queries.

We next compare the achievable quantized MSE exponent at a fixed rate  $R > 0$  for the four querying policies.

- Adaptive BZ algorithm:  $E_{q,BZ}^*(R) \geq E_{q,BZ}(R) := -\ln(1/2 + \sqrt{\epsilon(1-\epsilon)}) - R = E_0(1/2, \epsilon) - R$ .
- Non-adaptive UEP repetition querying:  $E_{q,repetition}^*(R) = 0$ .
- Non-adaptive block querying based on random block coding:  $E_{q,rc}^*(R) = E_r(R)$  for  $E_r(R)$  in (38), i.e.,

$$E_r(R) = \begin{cases} E_0(1/2, \epsilon) - R, & 0 \leq R < R_{\text{crit}}(\epsilon), \\ D_B(\gamma_{GV}(R) \parallel \epsilon), & R_{\text{crit}}(\epsilon) \leq R \leq C. \end{cases} \quad (75)$$

- Non-adaptive block querying based on superposition coding:  $E_{q,spc}^*(R) \geq E_{q,spc}(R)$  where

$$E_{q,spc}(R) = \begin{cases} E_0(1/2, \alpha * \epsilon) - R_1, & 0 \leq R_1 \leq R_{\text{crit}}(\alpha * \epsilon), \\ D_B(\gamma_{GV}(R_1) \parallel \alpha * \epsilon), & R_{\text{crit}}(\alpha * \epsilon) < R_1 \leq C - C_2(\alpha), \end{cases} \quad (76)$$

for  $\alpha = \alpha^*$  such that  $C_2(\alpha^*) = \frac{6}{5} (R - \frac{C}{6})$  and for  $R_1 = R - C_2(\alpha^*)$ .

Note that the exponent  $E_{q,BZ}(R)$  for the adaptive BZ algorithm and the exponent  $E_{q,spc}(R)$  for the non-adaptive block querying based on superposition coding are lower bounds on the best achievable exponents of each of the querying policies, respectively, while  $E_{q,repetition}^*(R)$  for the non-adaptive repetition querying and  $E_{q,rc}^*(R)$  for the

random block coding are the best achievable exponents of each of the policies. When we compare the four exponents  $E_{q,BZ}(R)$ ,  $E_{q,repetition}^*(R)$ ,  $E_{q,rc}^*(R)$ , and  $E_{q,spc}(R)$ , we can show that

$$E_{q,spc}(R) \geq E_{q,rc}^*(R) \geq E_{q,BZ}(R) \geq E_{q,repetition}^*(R), \quad (77)$$

which implies that the non-adaptive block querying based on the superposition code achieves the best, or at least as fast error rates of convergence for the quantized MSE compared to the two other non-adaptive policies including the block querying based on random block code and the UEP repetition querying. The first inequality between  $E_{q,spc}(R)$  and  $E_{q,rc}^*(R)$  follows from the fact that random block coding is a special case of superposition coding where the parameter  $\alpha$  equals  $1/2$ . In Theorem 1, we also demonstrated a strictly positive gain  $E_{q,spc}(R) > E_{q,rc}^*(R)$  in the high rate regimes of  $R$  for a very noisy BSC( $\epsilon$ ). The second inequality,  $E_{q,rc}^*(R) \geq E_{q,BZ}(R)$ , is true since  $E_{q,rc}^*(R) = E_{q,BZ}(R)$  in the low rate regime of  $0 \leq R \leq R_{crit}(\epsilon)$  while  $E_{q,rc}^*(R) > E_{q,BZ}(R)$  in the high rate regime of  $R_{crit}(\epsilon) < R \leq C$ . However, this does not necessarily mean that the non-adaptive querying based on random block coding outperforms the adaptive BZ algorithm since there may be better bounds than  $E_{q,BZ}(R)$  on the best achievable exponent  $E_{q,BZ}^*(R)$ . The last inequality holds since  $E_{q,repetition}^*(R) = 0$  for every rate  $R$ .

In Fig 10, we provide a plot of  $E_{q,spc}(R)$  (red dashed line) and  $E_{q,rc}^*(R)$  (black solid line) with the line  $2R$  (blue dash-dot line) for the BSC( $\epsilon$ ) with  $\epsilon = 0.45$ . We can observe that the lower bound  $E_{q,spc}(R)$  on the best achievable exponent of the superposition coding is strictly larger than the best achievable exponent  $E_{q,rc}^*(R)$  of the random block code in high rate regimes of  $R \in (C/6, C)$ . As stated in Corollary 1, the gain in the quantized MSE exponent in the high rate regimes of  $R \in (C/6, C)$  from superposition coding also results in a gain in the MSE exponent within this rate regime where  $2R \geq E_{q,spc}(R) \geq E_{q,rc}^*(R)$ .

## VI. CONCLUSION

We have considered estimation of the value of a continuous random variable in the context of the noisy 20 questions problem and proposed a non-adaptive block querying policy based on superposition coding that can provide unequal error protection for MSBs vs. LSBs of the binary expansion of the target variable, when the information bits are queried through a noisy BSC( $\epsilon$ ). Different from the UEP repetition code considered in [1] where the best achievable quantized MSE decreases exponentially only in  $\sqrt{N}$  where  $N$  is the number of queries, our non-adaptive querying policy based on superposition coding achieves exponential decrease in  $N$ , matching the rate of an adaptive 20 questions scheme. Moreover, the achievable MSE exponent is strictly better than that of a random block code at high rate regimes of  $R$  for a very noisy BSC( $\epsilon$ ), where  $R$  determines the resolution of the querying region as  $k = NR / \ln 2$  bits.

There are several open directions worthy of further study. First, while our lower bound in (56) on the best achievable quantized MSE exponent with the proposed UEP block querying demonstrates a strictly positive gain compared to that of the random block code in high rate regimes, it does not demonstrate any gain in low rate regimes. It would be interesting to see if a strictly positive gain can also be provided by the proposed UEP coding in the low rate regimes. It may require analysis of the error exponent of the more important partial message  $M_1$  (MSBs) with

the optimal ML decoder in order to demonstrate a gain in the error exponent of  $M_1$  from the superposition code even when  $R$  is small. Second, the proposed UEP querying policy can be generalized by using superposition coding with more than two levels of priority. This generalization may also improve the achievable estimation error since the resulting UEP querying policy would be able to use the querying resource more efficiently by providing finer levels of error protection for information bits of different significance, compared to the two level case we considered in this paper. Showing the improved performance, however, might require more complicated analysis of the error exponents of each of the partial messages. Third, the UEP block querying developed in this paper can be applied to state estimation problems with different cost functions other than  $L^2$ -norm considered in this paper. The UEP block querying policy based on superposition coding is a general approach to provide different error protection for two levels of priority. When the cost function is specified, the different importance of each of the partial message can be quantified by the weights on the partial decoding error probabilities in the total estimation error. These weights then determines the optimal choice of the parameters such as the rates of each of partial message as well as the distribution of the superposition code in order to minimize the total estimation error. Therefore, the UEP block querying strategy can be adopted for general state estimation problems by choosing the right parameters for the superposition code given the specified cost function.

## APPENDIX A

### PROOF OF PROPOSITION 1: ADAPTIVE BISECTION POLICY

Successive entropy minimization policies choose the most informative querying region  $Q_i$ , which queries 1 bit of information about the target variable  $X$  by balancing the probabilities

$$\Pr(X \in Q_i | Y_1^{i-1} = y_1^{i-1}) = \Pr(X \notin Q_i | Y_1^{i-1} = y_1^{i-1}) = 1/2 \quad (78)$$

for the collected answers  $y_1^{i-1} \in \{0, 1\}^{i-1}$  at the  $i$ -th querying round. For a continuous random variable  $X \sim p(x|y_1^{i-1})$ , there exist diverse ways to design such a querying region  $Q_i$  satisfying the condition (78).

We quantify the importance of the information bit about the target variable  $X \sim p(x|y_1^{i-1})$  asked at the  $i$ -th querying round by the predicted variance reduction, defined as

$$\text{Var}(X | Y_1^{i-1} = y_1^{i-1}) - \mathbb{E}[\text{Var}(X | Y_i, Y_1^{i-1} = y_1^{i-1})]. \quad (79)$$

Note that  $\mathbb{E}[\text{Var}(X | Y_i, Y_1^{i-1} = y_1^{i-1})]$  depends on the choice of  $Q_i$  since the two possible posterior distributions of  $X$ ,  $p(x|y_i = 0, y_1^{i-1})$  and  $p(x|y_i = 1, y_1^{i-1})$ , after the  $i$ -th querying, are functions of the choice of  $Q_i$ . The objective is to find the querying region  $Q_i$  that not only extracts 1 bit of information about  $X$  by satisfying (78) but also maximizes the predicted variance reduction (79), i.e., we aim to solve

$$\max_{\{Q_i: \Pr(X \in Q_i | Y_1^{i-1} = y_1^{i-1}) = \Pr(X \notin Q_i | Y_1^{i-1} = y_1^{i-1}) = 1/2\}} (\text{Var}(X | Y_1^{i-1} = y_1^{i-1}) - \mathbb{E}[\text{Var}(X | Y_i, Y_1^{i-1} = y_1^{i-1})]). \quad (80)$$

Proposition 1 states that the optimum  $Q_i$  that is the solution of the optimization (80) is the querying region that corresponds to the bisection policy, i.e., the optimum  $Q_i$  is right region of the median of  $p(x|y_1^{i-1})$ .

For adaptive sequential querying, given answers  $y_1^{i-1}$  of the previous queries  $(Q_1, \dots, Q_{i-1})$  the updated posterior distribution  $p(x|y_1^{i-1})$  specifies the distribution of  $X$  as  $q(x) = p(x|y_1^{i-1})$ . Define a one-step encoding map  $d : [0, 1] \rightarrow \{0, 1\}$  that maps the value of  $x \in [0, 1]$  to a binary bit  $Z = d(x)$ . This encoding map  $d(\cdot)$  is defined as the indicator function of  $Q$ :  $d(x) = 1$  for every  $x \in Q$  and  $d(x) = 0$  for every  $x \notin Q$ . We derive the optimum one-step encoding map  $d(\cdot)$  that maximally reduces the conditional variance of  $X \sim q(x)$  given the noisy answer  $Y = Z \oplus N$  where  $N \sim \text{Bernoulli}(\epsilon)$ .

The conditional variance of  $X$  given  $Y$  can be written as

$$\begin{aligned} \mathbb{E}[\text{Var}(X|Y)] &= \Pr(Y = 0)\text{Var}(X|Y = 0) + \Pr(Y = 1)\text{Var}(X|Y = 1) \\ &= \Pr(Y = 0) \left( \mathbb{E}[X^2|Y = 0] - (\mathbb{E}[X|Y = 0])^2 \right) + \Pr(Y = 1) \left( \mathbb{E}[X^2|Y = 1] - (\mathbb{E}[X|Y = 1])^2 \right) \\ &= \mathbb{E}[X^2] - \left( \Pr(Y = 0) (\mathbb{E}[X|Y = 0])^2 + \Pr(Y = 1) (\mathbb{E}[X|Y = 1])^2 \right). \end{aligned} \quad (81)$$

Since  $\mathbb{E}[X^2]$  does not depend on the encoding map  $d(\cdot)$ , to minimize  $\mathbb{E}[\text{Var}(X|Y)]$  the encoding map  $d(\cdot)$  should maximize

$$G_d := \Pr(Y = 0) (\mathbb{E}[X|Y = 0])^2 + \Pr(Y = 1) (\mathbb{E}[X|Y = 1])^2. \quad (82)$$

After applying Bayes' rule,  $G_d$  in (82) can be expressed in terms of the probabilities  $\{\Pr(Z = 0), \Pr(Z = 1)\}$  and the conditional expectations over the querying region  $\{\mathbb{E}[X|Z = 0] = \mathbb{E}[X|X \in Q], \mathbb{E}[X|Z = 1] = \mathbb{E}[X|X \notin Q]\}$  as

$$\begin{aligned} G_d &= \left( \frac{((1 - \epsilon) \cdot \Pr(Z = 0)\mathbb{E}[X|Z = 0] + \epsilon \cdot \Pr(Z = 1)\mathbb{E}[X|Z = 1])^2}{\Pr(Y = 0)} \right. \\ &\quad \left. + \frac{(\epsilon \cdot \Pr(Z = 0)\mathbb{E}[X|Z = 0] + (1 - \epsilon) \cdot \Pr(Z = 1)\mathbb{E}[X|Z = 1])^2}{\Pr(Y = 1)} \right). \end{aligned} \quad (83)$$

Note that both of  $\{\Pr(Z = 0), \Pr(Z = 1)\}$  and  $\{\mathbb{E}[X|Z = 0], \mathbb{E}[X|Z = 1]\}$  depend on the encoding map  $d(\cdot)$  since

$$\begin{aligned} \Pr(Z = 0) &= \Pr(X \notin Q_i) = \int_{d(x)=0} q(x)dx, \quad \Pr(Z = 1) = \Pr(X \in Q_i) = \int_{d(x)=1} q(x)dx, \\ \mathbb{E}[X|Z = a] &= \frac{1}{\Pr(Z = a)} \int_{d(x)=a} xq(x)dx, \quad \text{for } a = 0, 1. \end{aligned} \quad (84)$$

When  $m$  denotes the mean of  $X \sim q(x)$ ,  $\{\Pr(Z = 0), \Pr(Z = 1)\}$  and  $\{\mathbb{E}[X|Z = 0], \mathbb{E}[X|Z = 1]\}$  should satisfy

$$m = \mathbb{E}[X] = \Pr(Z = 0)\mathbb{E}[X|Z = 0] + \Pr(Z = 1)\mathbb{E}[X|Z = 1]. \quad (85)$$

We show that among encoding maps  $d(\cdot)$  having a fixed  $\{\Pr(Z = 0), \Pr(Z = 1)\}$  the encoding map that maximizes  $G_d$  in (83), which thus minimizes  $\mathbb{E}[\text{Var}(X|Y)]$ , under the constraint of (85) should be a step function.

*Lemma 7: Among encoding maps  $d : [0, 1] \rightarrow \{0, 1\}$  having the same  $\{\Pr(Z = 0), \Pr(Z = 1)\}$ , the optimum encoding map that maximizes  $G_d$  in (82) under the constraint of (85) should maximize  $|\mathbb{E}[X|Z = 0] - m|$ . Therefore, for a fixed  $\Pr(Z = 0)$ , in order to maximize  $|\mathbb{E}[X|Z = 0] - m|$  the optimum encoding map  $d : [0, 1] \rightarrow \{0, 1\}$  should be either*

$$d(x) = \begin{cases} 0 & x \leq t_1 \\ 1 & x > t_1 \end{cases} \quad (86)$$



for the threshold  $t_1$  such that  $\int_0^{t_1} q(x)dx = \alpha$ , or

$$d(x) = \begin{cases} 1 & x \leq t_2 \\ 0 & x > t_2 \end{cases} \quad (87)$$

for the threshold  $t_2$  such that  $\int_{t_2}^1 q(x)dx = \alpha$ .

*Proof:* Let us define  $\beta := \Pr(Z = 0)$  and  $A_\beta := \mathbb{E}[X|Z = 0]$  for a fixed  $0 \leq \beta \leq 1$ . From the constraints in (85), it becomes  $\Pr(Z = 1)\mathbb{E}[X|Z = 1] = m - \beta A_\beta$ . By using these parameters, we can rewrite  $G_d$  in (83) as

$$\begin{aligned} G_d &= \frac{((1-\epsilon)\beta A_\beta + \epsilon(m - \beta A_\beta))^2}{\epsilon + (1-2\epsilon)\beta} + \frac{(\epsilon\beta A_\beta + (1-\epsilon)(m - \beta A_\beta))^2}{(1-\epsilon) - (1-2\epsilon)\beta} \\ &= \frac{((1-\epsilon)\beta A_\beta + \epsilon(m - \beta A_\beta))^2 ((1-\epsilon) - (1-2\epsilon)\beta) + (\epsilon\beta A_\beta + (1-\epsilon)(m - \beta A_\beta))^2 (\epsilon + (1-2\epsilon)\beta)}{(\epsilon + (1-2\epsilon)\beta)((1-\epsilon) - (1-2\epsilon)\beta)}. \end{aligned} \quad (88)$$

By rearranging the terms, the numerator of  $G_d$  can be simplified as

$$\begin{aligned} &((1-\epsilon)^2 \beta^2 A_\beta^2 + 2\epsilon(1-\epsilon)\beta A_\beta(m - \beta A_\beta) + \epsilon^2(m - \beta A_\beta)^2)((1-\epsilon) - (1-2\epsilon)\beta) \\ &+ (\epsilon^2 \beta^2 A_\beta^2 + 2\epsilon(1-\epsilon)\beta A_\beta(m - \beta A_\beta) + (1-\epsilon)^2(m - \beta A_\beta)^2)(\epsilon + (1-2\epsilon)\beta) \\ &= (1-2\epsilon)^2 \beta^2 (A_\beta - m)^2 + m^2(\epsilon(1-\epsilon) + (1-2\epsilon)^2 \beta(1-\beta)). \end{aligned} \quad (89)$$

By using this simplified numerator,  $G_d$  can be written as

$$\begin{aligned} G_d &= \frac{(1-2\epsilon)^2 \beta^2 (A_\beta - m)^2 + m^2(\epsilon(1-\epsilon) + (1-2\epsilon)^2 \beta(1-\beta))}{\epsilon(1-\epsilon) + (1-2\epsilon)^2 \beta(1-\beta)} \\ &= \frac{(1-2\epsilon)^2 \beta^2 (A_\beta - m)^2}{\epsilon(1-\epsilon) + (1-2\epsilon)^2 \beta(1-\beta)} + m^2. \end{aligned} \quad (90)$$

Therefore, for a fixed  $(\beta, \epsilon, m)$ ,  $G_d$  is maximized when  $A_\beta$  is as far as possible from the mean  $m = \mathbb{E}[X]$  of  $X \sim q(x)$ . For a fixed  $\beta = \Pr(Z = 0)$ , the optimum encoding map  $d : [0, 1] \rightarrow \{0, 1\}$  that maximizes  $|A_\beta - m| = |\mathbb{E}[X|Z = 0] - m|$  should be a step function of either (86) or (87).  $\blacksquare$

For successive entropy minimization strategies that query 1 bit of information about  $X \sim q(x)$  at each round, the probabilities of the event  $\{Z = 0\}$  and of  $\{Z = 1\}$  are balanced as

$$\Pr(Z = 0) = \Pr(Z = 1) = 1/2. \quad (91)$$

The thresholds of the two step functions (86) and (87) for this case become the same as the median of the distribution. Lemma 7 thus implies that among policies satisfying (91), the adaptive bisection policy is the optimum one-step policy that minimizes the conditional variance of  $X$  given a noisy answer  $Y$ .

## APPENDIX B

### PROOF OF LEMMA 2: BIT ERROR PROBABILITY WITH REPETITION CODING

In Lemma 2, we show that when a binary bit  $B_i \sim \text{Bernoulli}(1/2)$  is repeatedly transmitted through a BSC( $\epsilon$ ) by  $N_i$  times, the decoding error probability of  $B_i$  with the majority voting algorithm is bounded below and above as

$$\frac{e^{-1/(3N_i)}}{\sqrt{2\pi N_i}} e^{-N_i D_B(1/2\|\epsilon)} \leq \Pr(\hat{B}_i \neq B_i) \leq e^{-N_i D_B(1/2\|\epsilon)}. \quad (92)$$

We first prove the upper bound. When the  $N_i$  channel outputs for the input  $B_i$  are denoted as  $(Y_1, \dots, Y_{N_i}) \in \{0, 1\}^{N_i}$ , define  $Y'_j = 2Y_j - 1 \in \{-1, 1\}$ . The majority voting claims an estimate  $\hat{B}_i = 1$  when  $\sum_{j=1}^{N_i} Y'_j > 0$ , and  $\hat{B}_i = 0$  when  $\sum_{j=1}^{N_i} Y'_j \leq 0$ . Since the channel is symmetric and  $B_i \sim \text{Bernoulli}(1/2)$ , the bit error probability can be bounded above as

$$\begin{aligned} \Pr(\hat{B}_i \neq B_i) &= \frac{1}{2} \Pr\left(\sum_{j=1}^{N_i} Y'_j > 0 | B_i = 0\right) + \frac{1}{2} \Pr\left(\sum_{j=1}^{N_i} Y'_j \leq 0 | B_i = 1\right) \\ &\leq \Pr\left(\sum_{j=1}^{N_i} Y'_j \leq 0 | B_i = 1\right) \\ &= \Pr\left(e^{-\lambda \sum_{j=1}^{N_i} Y'_j} \geq 1 | B_i = 1\right), \text{ for } \lambda > 0 \\ &\leq \mathbb{E}\left[e^{-\lambda \sum_{j=1}^{N_i} Y'_j} | B_i = 1\right], \end{aligned} \quad (93)$$

where the last inequality is from the Markov's inequality.

Using the conditional independence of  $Y'_1, \dots, Y'_{N_i}$  given  $B_i = 1$  and the fact that  $\Pr(Y'_j = 1 | B_i = 1) = 1 - \epsilon$ ,  $\Pr(Y'_j = -1 | B_i = 1) = \epsilon$ ,

$$\begin{aligned} \mathbb{E}\left[e^{-\lambda \sum_{j=1}^{N_i} Y'_j} | B_i = 1\right] &= \prod_{j=1}^{N_i} \mathbb{E}\left[e^{-\lambda Y'_j} | B_i = 1\right] \\ &= \prod_{j=1}^{N_i} ((1 - \epsilon)e^{-\lambda} + \epsilon \cdot e^{\lambda}). \end{aligned} \quad (94)$$

When  $\lambda = \log \sqrt{\frac{1-\epsilon}{\epsilon}}$ , the term  $((1 - \epsilon)e^{-\lambda} + \epsilon \cdot e^{\lambda})$  is minimized as  $2\sqrt{\epsilon(1 - \epsilon)} = e^{-D_B(1/2\|\epsilon)}$ . Therefore, the bit error probability can be bounded above as

$$\Pr(\hat{B}_i \neq B_i) \leq \min_{\lambda} \left( \prod_{j=1}^{N_i} ((1 - \epsilon)e^{-\lambda} + \epsilon \cdot e^{\lambda}) \right) = e^{-N_i D_B(1/2\|\epsilon)}. \quad (95)$$

We next prove the lower bound. The bit error probability can be bounded below as

$$\Pr(\hat{B}_i \neq B_i) \geq \frac{1}{2} \Pr\left(\sum_{j=1}^{N_i} Y'_j = 0 | B_i = 1\right). \quad (96)$$

The probability  $\Pr\left(\sum_{j=1}^{N_i} Y'_j = 0 | B_i = 1\right)$  is the probability of the event that half of the transmitted bits are flipped by the channel noise of Bern(1/2) distribution. This probability is bounded below as

$$\begin{aligned} \Pr\left(\sum_{j=1}^{N_i} Y'_j = 0 | B_i = 1\right) &= \binom{N_i}{N_i/2} \epsilon^{N_i/2} (1 - \epsilon)^{N_i/2} \\ &\geq \sqrt{\frac{2}{\pi N_i}} e^{-1/(3N_i)} 2^N \epsilon^{N_i/2} (1 - \epsilon)^{N_i/2} \\ &= \sqrt{\frac{2}{\pi N_i}} e^{-1/(3N_i)} e^{-N_i D_B(1/2\|\epsilon)} \end{aligned} \quad (97)$$

where the middle inequality is from the Stirling bound. By plugging this lower bound to (96), we obtain

$$\Pr(\hat{B}_i \neq B_i) \geq \frac{e^{-1/(3N_i)}}{\sqrt{2\pi N_i}} e^{-N_i D_B(1/2\|\epsilon)}. \quad (98)$$

## APPENDIX C

## PROOF OF LEMMA 4: QUANTIZED MSE EXPONENT WITH RANDOM BLOCK CODING

In Lemma 4, we show that the best achievable quantized MSE exponent with the random block code is

$$E_{q,rc}^*(R) = E_r(R). \quad (99)$$

The achievability of (99) was shown by the bound (40). In this section we prove the converse, i.e., the quantized MSE exponent with the random block code cannot be better than  $E_r(R)$ . We prove this by providing a lower bound on the quantized MSE.

Consider the quantized MSE expanded in terms of the conditional bit error probabilities.

$$\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})] = \sum_{i=1}^k \Pr(\hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}) \mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}}) | \hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}]. \quad (100)$$

Since  $\Pr(\hat{M} = M) \rightarrow 1$  as  $N \rightarrow \infty$  for the random block code of rate  $R \in (0, C)$ , we know that  $\Pr(\hat{B}_1^i = B_1^i) \doteq 1$  for any  $i = 1, \dots, k$ . Therefore, we can write  $\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})]$  as

$$\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})] \doteq \sum_{i=1}^k \Pr(\hat{B}_i \neq B_i | \hat{B}_1^{i-1} = B_1^{i-1}) \mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}}) | \hat{B}_i \neq B_i, \hat{B}_1^{i-1} = B_1^{i-1}]. \quad (101)$$

We then show that the first term of the summation of the right hand side is

$$\Pr(\hat{B}_1 \neq B_1) \mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}}) | \hat{B}_1 \neq B_1] \stackrel{\geq}{\geq} e^{-NE_r(R)}. \quad (102)$$

for the random coding error exponent  $E_r(R)$ .

Note that

$$\Pr(\hat{B}_1 \neq B_1) \leq \Pr(\hat{M} \neq M) \leq \sum_{i=1}^k \Pr(\hat{B}_i \neq B_i). \quad (103)$$

The average bit error probability  $\Pr(\hat{B}_i \neq B_i)$  for the random block code is the same for every  $i \in \{1, \dots, k\}$  from the symmetry among the information bits  $B_i$ 's in encoding. Since  $k = NR / \ln 2$  increases linearly in  $N$ , the exponent of  $\Pr(\hat{B}_1 \neq B_1)$  is the same as that of  $\Pr(\hat{M} \neq M) \doteq e^{-NE_r(R)}$ , i.e.,  $\Pr(\hat{B}_1 \neq B_1) \doteq e^{-NE_r(R)}$ . Moreover,

$$\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}}) | \hat{B}_1 \neq B_1] \doteq 1. \quad (104)$$

This can be shown by calculating a lower bound on  $\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}}) | \hat{B}_1 \neq B_1]$ . When  $\hat{B}_1 = 1$  and  $B_1 = 0$ , the best  $\hat{X}_N$  that minimizes the conditional expectation is  $\hat{X}_N = 1/2$ . Conditioned on  $B_1 = 0$ ,  $X$  is uniformly distributed over  $[0, 1/2]$ , and thus

$$\mathbb{E}[c_q(X, \hat{X}_N) | \hat{B}_1 = 1, B_1 = 0] \geq \int_0^{1/2} 2(x - 1/2)^2 dx = 1/12. \quad (105)$$

The same bound also holds when  $\hat{B}_1 = 0$  and  $B_1 = 1$ . From this lower bound on  $\mathbb{E}[c_q(X, \hat{X}_N) | \hat{B}_1 = 1, B_1 = 0]$  and the bit error probability  $\Pr(\hat{B}_1 \neq B_1) \doteq e^{-NE_r(R)}$ , the lower bound in (102) can be proven. From the bound (102) and the expansion on  $\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})]$  in (101), we can conclude that

$$\mathbb{E}[c_q(X, \hat{X}_{N,\text{finite}})] \stackrel{\geq}{\geq} e^{-NE_r(R)}. \quad (106)$$

## APPENDIX D

## PROOFS OF LEMMA 5: DECODING OF SUPERPOSITION CODES

In Lemma 5, we show that the superposition code provides a better, or at least as good error protection for the more important partial message  $M_1$  (MSBs) as that of the random block code by providing two lower bounds  $E_{\text{MSBs,JML}}(R_1, R_2)$  and  $E_{\text{MSBs,SC}}(R_1, \alpha)$  on the best achievable error exponent  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$ .

The first lower bound  $E_{\text{MSBs,JML}}(R_1, R_2)$  is defined as the best achievable error exponent for  $\Pr(\hat{M}_1 \neq M_1)$  with the joint maximum likelihood (JML) decoding rule. We show that  $E_{\text{MSBs,JML}}(R_1, R_2) \geq E_r(R_1 + R_2)$  for every  $(R_1, R_2, \alpha)$  in the following lemma.

*Lemma 8: For a given  $(R_1, R_2, \alpha)$ , the decoding error probability of  $M_1$  with the Joint ML decoding rule is bounded by*

$$\Pr(\hat{M}_{1,\text{JML}} \neq M_1) \leq e^{-N E_{\text{MSBs,JML}}^{\text{LB}}(R_1, R_2)}, \quad (107)$$

where

$$E_{\text{MSBs,JML}}^{\text{LB}}(R_1, R_2) = \begin{cases} E_0(1/2, \epsilon) - R_2 - R_1, & 0 \leq R_1 \leq \max\{0, R_{\text{crit}}(\epsilon) - R_2\}, \\ D_{\text{B}}(\gamma_{\text{GV}}(R_1 + R_2) \parallel \epsilon), & \max\{0, R_{\text{crit}}(\epsilon) - R_2\} < R_1 \leq H_{\text{B}}(1/2) - H_{\text{B}}(\epsilon) - R_2. \end{cases} \quad (108)$$

*Proof:* Appendix F ■

Note that for any given  $(R_1, R_2, \alpha)$  the achievable exponent  $E_{\text{MSBs,JML}}^{\text{LB}}(R_1, R_2)$  is equal to  $E_r(R_1 + R_2)$ . Since joint maximum likelihood decoding is a sub-optimal decoding rule, the fact that  $E_{\text{MSBs,JML}}(R_1, R_2) \geq E_r(R_1 + R_2)$  implies that the superposition code provides a better, or at least as good, error protection for the partial message  $M_1$  as that of the random block code for every  $(R_1, R_2)$ , regardless of the choice of  $\alpha \in (0, 1/2)$ .

We next prove a strict gain in the error exponent  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$  from the superposition coding by providing another lower bound  $E_{\text{MSBs,SC}}(R_1, \alpha)$  on  $E_{\text{MSBs}}^*(R_1, R_2, \alpha)$ , which is the best achievable error exponent with the successive cancellation (SC) decoding rules. As shown in Eq. (51), the exponent can be written as

$$E_{\text{MSBs,SC}}(R_1, \alpha) = \begin{cases} E_0(1/2, \alpha * \epsilon) - R_1, & 0 \leq R_1 \leq R_{\text{crit}}(\alpha * \epsilon), \\ D_{\text{B}}(\gamma_{\text{GV}}(R_1) \parallel \alpha * \epsilon), & R_{\text{crit}}(\alpha * \epsilon) < R_1 \leq C - C_2(\alpha), \end{cases} \quad (109)$$

with  $E_0(a, b) = -\ln(1 - 2a(1 - a)(\sqrt{b} - \sqrt{1 - b})^2)$  and thus  $E_0(1/2, \alpha * \epsilon) = -\ln(1/2 + \sqrt{(\alpha * \epsilon)(1 - (\alpha * \epsilon))})$ ,  $C = H_{\text{B}}(1/2) - H_{\text{B}}(\epsilon)$ ,  $C_2(\alpha) = H_{\text{B}}(\alpha * \epsilon) - H_{\text{B}}(\epsilon)$ ,  $R_{\text{crit}}(\alpha * \epsilon) = D_{\text{B}}(\gamma_{\text{crit}}(\alpha * \epsilon) \parallel 1/2)$  and  $\gamma_{\text{crit}}(\alpha * \epsilon) = \frac{\sqrt{\alpha * \epsilon}}{\sqrt{\alpha * \epsilon} + \sqrt{1 - \alpha * \epsilon}}$ .

By comparing  $E_{\text{MSBs,SC}}(R_1, \alpha)$  in (109) with the random block code exponent  $E_r(R_1 + R_2)$  in (38), we can show that for sufficiently large  $R_2$  and sufficiently small  $R_1$  the superposition code provides a strictly better error protection for the partial message  $M_1$  than does the random block code. More precisely, for a given  $\alpha \in (0, 1/2)$  and a fixed  $\epsilon \in (0, 1/2)$ , we can find thresholds  $R_{2,\text{th}}(\alpha, \epsilon)$  on  $R_2$  and  $R_{1,\text{th}}(R_2, \alpha, \epsilon)$  on  $R_1$  such that for  $R_{2,\text{th}}(\alpha, \epsilon) < R_2 < C_2(\alpha)$  and  $0 \leq R_1 < R_{1,\text{th}}(R_2, \alpha, \epsilon)$

$$E_{\text{MSBs,SC}}(R_1, \alpha) > E_r(R_1 + R_2). \quad (110)$$

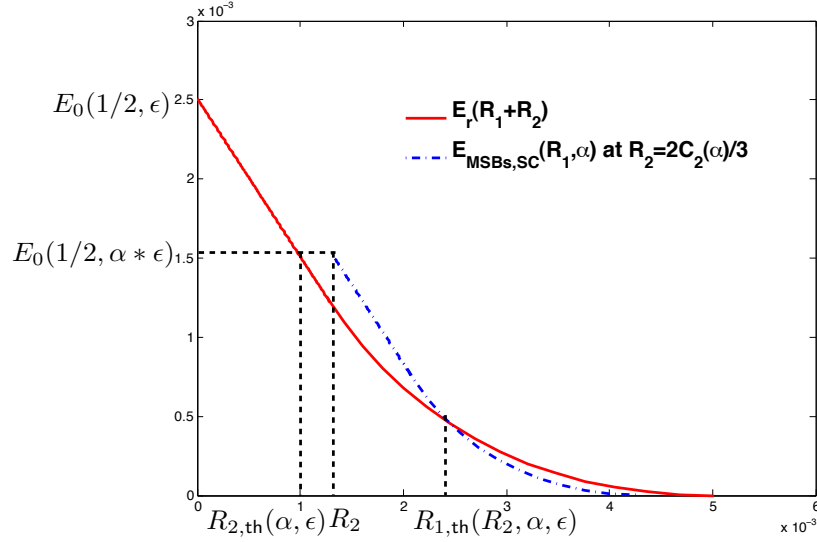


Fig. 11. A plot of  $E_r(R_1 + R_2)$  and  $E_{\text{MSBs,SC}}(R_1, \alpha)$  for  $\epsilon = 0.45$ ,  $\alpha = 0.11$ , and  $R_2 = 2C_2(\alpha)/3$ . The superposition code error exponent  $E_{\text{MSBs,SC}}(R_1, \alpha)$  for the partial message  $M_1$  is strictly greater than that of the random block code  $E_r(R_1, R_2)$  over  $R_1 \in (0, R_{1,\text{th}}(R_2, \alpha, \epsilon))$  for a fixed  $R_2 = 2C_2(\alpha)/3 > R_{2,\text{th}}(\alpha, \epsilon)$ .

The threshold on  $R_2$ , denoted  $R_{2,\text{th}}(\alpha, \epsilon)$ , is defined as the rate  $R$  at which the random coding error exponent  $E_r(R)$  equals  $E_{\text{MSBs,SC}}(0, \alpha) = E_0(1/2, \alpha * \epsilon)$ . The threshold on  $R_1$ , denoted  $R_{1,\text{th}}(R_2, \alpha, \epsilon)$ , is defined as the rate  $R_1$  such that  $E_{\text{MSBs,SC}}(R_1, \alpha) = E_r(R_1 + R_2)$  for a given rate  $R_2 > R_{2,\text{th}}(\alpha, \epsilon)$ . In Fig. 11, we plot  $E_{\text{MSBs,SC}}(R_1, \alpha)$  (blue dash-dot line) and  $E_r(R_1 + R_2)$  (red solid line) for  $\alpha = 0.11$ ,  $\epsilon = 0.45$  and  $R_2 = 2C_2(\alpha)/3 > R_{2,\text{th}}(\alpha, \epsilon)$ . From the plots, we can observe a gain in the error exponent of  $\Pr(\hat{M}_1 \neq M_1)$  for  $0 \leq R_1 \leq R_{1,\text{th}}(R_2, \alpha, \epsilon)$  at a fixed  $R_2 = 2C_2(\alpha)/3 > R_{2,\text{th}}(\alpha, \epsilon)$ . From the plots, we can also check the thresholds  $R_{2,\text{th}}(\alpha, \epsilon)$  and  $R_{1,\text{th}}(R_2, \alpha, \epsilon)$  for this case.

Lastly, we state a lower bound on the exponent  $E_{\text{LSBs,SC}}(R_2, \alpha)$  of  $\Pr(\hat{M}_2 \neq M_2 | \hat{M}_1 = M_1)$  with the SC decoding rule provided in [19]:

$$E_{\text{LSBs,SC}}(R_2, \alpha) := \max_{0 \leq \rho \leq 1} [F_0(\rho, \alpha) - \rho R_2] \quad (111)$$

where

$$F_0(\rho, \alpha) = -\ln \left\{ \sum_{y \in \{0,1\}} \left[ \sum_{v \in \{0,1\}} P_V(v) P_{Y|V}(y|v)^{\frac{1}{1+\rho}} \right]^{1+\rho} \right\} \quad (112)$$

for  $P_V(v)$  being the Bernoulli( $\alpha$ ) and  $P_{Y|V}(y|v)$  being the transition probability of BSC( $\epsilon$ ). We can show that  $E_{m_2}(R_2, \alpha) > 0$  for  $R_2 < C_2(\alpha)$  where  $C_2(\alpha)$  is equal to

$$C_2(\alpha) = \left. \frac{\partial F_0(\rho, \alpha)}{\partial \rho} \right|_{\rho=0} = H_B(\alpha * \epsilon) - H_B(\epsilon). \quad (113)$$

Note that as  $\alpha$  decreases from  $1/2$  to  $0$ , the maximum rate  $C_2(\alpha)$  of LSBs achieving a reliable communication keeps decreasing.

## APPENDIX E

PROOF OF LEMMA 6: GAIN IN THE ERROR EXPONENT OF MSBs WITH SUPERPOSITION CODING FOR A VERY NOISY BSC( $\epsilon$ )

We show that for a very noisy BSC( $\epsilon$ ) the superposition codes achieve a strictly positive gain in the error exponent of  $\Pr(\hat{M}_1 \neq M_1)$  for every  $R_1 \in (0, H_B(1/2) - H_B(\alpha * \epsilon))$  compared to that of the random block code when the rate  $R_2$  of the less important message  $M_2$  equals the maximum rate  $C_2(\alpha) = H_B(\alpha * \epsilon) - H_B(\alpha)$ .

When we fix  $R_2 = C_2(\alpha)$ , the best achievable  $\Pr(\hat{M}_1 \neq M_1)$  of the random block code with the rate  $R_1$  of the more important message  $M_1$  is

$$\Pr(\hat{M}_1 \neq M_1) \doteq e^{-NE_r(R_1 + R_2)} \quad (114)$$

where

$$E_r(R_1 + R_2) = \begin{cases} E_0(1/2, \epsilon) - C_2(\alpha) - R_1, & 0 \leq R_1 \leq \max\{0, R_{\text{crit}}(\epsilon) - C_2(\alpha)\}, \\ D_B(\gamma_{\text{GV}}(R_1 + C_2(\alpha))\|\epsilon), & \max\{0, R_{\text{crit}}(\epsilon) - C_2(\alpha)\} < R_1 \leq H_B(1/2) - H_B(\alpha * \epsilon), \end{cases} \quad (115)$$

for  $E_0(1/2, \epsilon) = \log 2 - \log(1 + 2\sqrt{\epsilon(1-\epsilon)})$ ,  $C_2(\alpha) = H_B(\alpha * \epsilon) - H_B(\epsilon)$  and  $R_{\text{crit}}(\epsilon) = D_B(\gamma_{\text{crit}}(\epsilon)\|1/2)$  where  $\gamma_{\text{crit}}(\epsilon) = \frac{\sqrt{\epsilon}}{\sqrt{\epsilon} + \sqrt{1-\epsilon}}$ .

With the superposition codes and the successive cancellation decoding, we can achieve

$$\Pr(\hat{M}_1 \neq M_1) \dot{\leq} e^{-NE_{\text{MSBs,SC}}(R_1, \alpha)} \quad (116)$$

where

$$E_{\text{MSBs,SC}}(R_1, \alpha) = \begin{cases} E_0(1/2, \alpha * \epsilon) - R_1, & 0 \leq R_1 \leq R_{\text{crit}}(\alpha * \epsilon), \\ D_B(\gamma_{\text{GV}}(R_1)\|\alpha * \epsilon), & R_{\text{crit}}(\alpha * \epsilon) < R_1 \leq H_B(1/2) - H_B(\alpha * \epsilon). \end{cases} \quad (117)$$

To prove that  $E_{\text{MSBs,SC}}(R_1, \alpha) > E_r(R_1 + R_2)$  for every  $R_1 \in [0, H_B(1/2) - H_B(\alpha * \epsilon)]$  at  $R_2 = C_2(\alpha)$ , we need to demonstrate the following three statements for  $\epsilon \approx 0.5$  at every  $\alpha \in (0, 1/2)$ ,

- 1)  $E_0(1/2, \alpha * \epsilon) > E_0(1/2, \epsilon) - C_2(\alpha)$ .
- 2)  $E_0(1/2, \alpha * \epsilon) - R_1 > D_B(\gamma_{\text{GV}}(R_1 + C_2(\alpha))\|\epsilon)$  for  $0 \leq R_1 \leq R_{\text{crit}}(\alpha * \epsilon)$  when  $R_{\text{crit}}(\epsilon) < C_2(\alpha)$ .
- 3)  $D_B(\gamma_{\text{GV}}(R_1)\|\alpha * \epsilon) > D_B(\gamma_{\text{GV}}(R_1 + C_2(\alpha))\|\epsilon)$  for  $R_{\text{crit}}(\alpha * \epsilon) < R_1 < H_B(1/2) - H_B(\alpha * \epsilon)$ .

Once these three statements are proven, it is sufficient to show that  $E_{\text{MSBs,SC}}(R_1, \alpha) > E_r(R_1 + R_2)$  for every  $0 \leq R_1 \leq H_B(1/2) - H_B(\alpha * \epsilon)$  when  $R_2 = C_2(\alpha)$ .

The first statement 1)  $E_0(1/2, \alpha * \epsilon) > E_0(1/2, \epsilon) - C_2(\alpha)$  is equivalent to

$$H_B(\alpha * \epsilon) - \log(\sqrt{\alpha * \epsilon} + \sqrt{1 - \alpha * \epsilon})^2 > H_B(\epsilon) - \log(\sqrt{\epsilon} + \sqrt{1 - \epsilon})^2. \quad (118)$$

When we define  $f(x) = H_B(x) - \log(\sqrt{x} + \sqrt{1-x})^2$ , the above inequality is equivalent to  $f(\alpha * \epsilon) - f(\epsilon) > 0$ . Note that  $0 \leq \epsilon < \alpha * \epsilon \leq 1/2$  for every  $\alpha \in (0, 1/2)$ . It can be checked that the derivative of  $f(x)$  is positive in the regime of  $0.05 \leq x \leq 1/2$ . Therefore, for a very noisy channel with  $\epsilon \geq 0.05$ , the statement 1)  $E_0(1/2, \alpha * \epsilon) > E_0(1/2, \epsilon) - C_2(\alpha)$  is true.

We next prove the statement 3)  $D_B(\gamma_{GV}(R_1)\|\alpha * \epsilon) > D_B(\gamma_{GV}(R_1 + C_2(\alpha))\|\epsilon)$  for  $R_{\text{crit}}(\alpha * \epsilon) < R_1 < H_B(1/2) - H_B(\alpha * \epsilon)$ , which will also be used to prove the statement 2). First, note that at  $R_1 = H_B(1/2) - H_B(\alpha * \epsilon)$ ,  $D_B(\gamma_{GV}(R_1)\|\alpha * \epsilon) = D_B(\gamma_{GV}(R_1 + C_2(\alpha))\|\epsilon) = 0$  from the definition of  $\gamma_{GV}(R)$ . We will prove that  $D_B(\gamma_{GV}(R_1)\|\alpha * \epsilon) - D_B(\gamma_{GV}(R_1 + C_2(\alpha))\|\epsilon)$  strictly decreases in  $R_1 \in (R_{\text{crit}}(\alpha * \epsilon), H_B(1/2) - H_B(\alpha * \epsilon)]$ . Since  $D_B(\gamma_{GV}(R_1)\|\alpha * \epsilon) = D_B(\gamma_{GV}(R_1 + C_2(\alpha))\|\epsilon) = 0$  at  $R_1 = H_B(1/2) - H_B(\alpha * \epsilon)$ , the fact that the difference between the two divergences strictly decreases implies the statement 3). To show that  $D_B(\gamma_{GV}(R_1)\|\alpha * \epsilon) - D_B(\gamma_{GV}(R_1 + C_2(\alpha))\|\epsilon)$  keeps decreasing, we will prove

$$\frac{\partial}{\partial R_1} D_B(\gamma_{GV}(R_1)\|\alpha * \epsilon) < \frac{\partial}{\partial R_1} D_B(\gamma_{GV}(R_1 + C_2(\alpha))\|\epsilon). \quad (119)$$

From the definition of  $\gamma_{GV}(R)$ , it satisfies  $\ln 2 + \gamma_{GV}(R) \ln \gamma_{GV}(R) + (1 - \gamma_{GV}(R)) \ln(1 - \gamma_{GV}(R)) = R$ . By differentiating both sides by  $\gamma_{GV}(R)$  and re-arranging the terms, we get

$$\frac{\partial \gamma_{GV}(R)}{\partial R} = -\frac{1}{\log \frac{1-\gamma_{GV}(R)}{\gamma_{GV}(R)}}. \quad (120)$$

From  $\frac{\partial}{\partial x} D_B(x\|y) = \log \left( \frac{x}{1-x} \frac{1-y}{y} \right)$  and (120), we have

$$\begin{aligned} \frac{\partial}{\partial R_1} D_B(\gamma_{GV}(R_1)\|\alpha * \epsilon) &= 1 - \frac{\log \frac{1-\alpha * \epsilon}{\alpha * \epsilon}}{\log \frac{1-\gamma_{GV}(R_1)}{\gamma_{GV}(R_1)}}, \\ \frac{\partial}{\partial R_1} D_B(\gamma_{GV}(R_1 + C_2(\alpha))\|\epsilon) &= 1 - \frac{\log \frac{1-\epsilon}{\epsilon}}{\log \frac{1-\gamma_{GV}(R_1 + C_2(\alpha))}{\gamma_{GV}(R_1 + C_2(\alpha))}}. \end{aligned} \quad (121)$$

Therefore, showing (119) is equivalent to showing

$$\frac{\log \frac{1-\epsilon}{\epsilon}}{\log \frac{1-\alpha * \epsilon}{\alpha * \epsilon}} < \frac{\log \frac{1-\gamma_{GV}(R_1 + C_2(\alpha))}{\gamma_{GV}(R_1 + C_2(\alpha))}}{\log \frac{1-\gamma_{GV}(R_1)}{\gamma_{GV}(R_1)}}. \quad (122)$$

To prove this inequality, we will first show that

$$\gamma_{GV}(R_1) - \gamma_{GV}(R_1 + C_2(\alpha)) \geq \alpha * \epsilon - \epsilon \quad (123)$$

Note that  $\alpha * \epsilon = \gamma_{GV}(H_B(1/2) - H_B(\alpha * \epsilon))$ ,  $\epsilon = \gamma_{GV}(H_B(1/2) - H_B(\epsilon))$ . Therefore, (123) can be written as

$$\gamma_{GV}(R_1) - \gamma_{GV}(R_1 + C_2(\alpha)) \geq \gamma_{GV}(H_B(1/2) - H_B(\alpha * \epsilon)) - \gamma_{GV}(H_B(1/2) - H_B(\epsilon)). \quad (124)$$

Note that  $\gamma_{GV}(R)$  is convex and decreasing in  $R$ . Moreover, we know that  $(H_B(1/2) - H_B(\epsilon)) - (H_B(1/2) - H_B(\alpha * \epsilon)) = C_2(\alpha)$ . Since we consider the regime where  $R_1 \leq H_B(1/2) - H_B(\alpha * \epsilon)$ , from the convexity of  $\gamma_{GV}(R)$ , the inequality in (124) can be implied. Therefore, for  $c := \alpha * \epsilon - \epsilon$ ,  $\gamma_{GV}(R_1 + C_2(\alpha)) \leq \gamma_{GV}(R_1) - c \leq 1/2$ .

Since  $\log \frac{1-x}{x}$  is decreasing in  $0 \leq x \leq 1/2$  and  $\gamma_{GV}(R_1 + C_2(\alpha)) \leq \gamma_{GV}(R_1) - c \leq 1/2$ , we have

$$\frac{\log \frac{1-(\gamma_{GV}(R_1)-c)}{(\gamma_{GV}(R_1)-c)}}{\log \frac{1-\gamma_{GV}(R_1)}{\gamma_{GV}(R_1)}} \leq \frac{\log \frac{1-\gamma_{GV}(R_1+C_2(\alpha))}{\gamma_{GV}(R_1+C_2(\alpha))}}{\log \frac{1-\gamma_{GV}(R_1)}{\gamma_{GV}(R_1)}} \quad (125)$$

Therefore, to prove (122), it is sufficient to show that

$$\frac{\log \frac{1-\epsilon}{\epsilon}}{\log \frac{1-\alpha * \epsilon}{\alpha * \epsilon}} < \frac{\log \frac{1-(\gamma_{GV}(R_1)-c)}{(\gamma_{GV}(R_1)-c)}}{\log \frac{1-\gamma_{GV}(R_1)}{\gamma_{GV}(R_1)}}. \quad (126)$$

Since  $\alpha * \epsilon < \gamma_{\text{GV}}(R_1)$  and  $c = \alpha * \epsilon - \epsilon = \gamma_{\text{GV}}(R_1) - (\gamma_{\text{GV}}(R_1) - c)$ , if we can prove that

$$\frac{\log \frac{1-(x-c)}{(x-c)}}{\log \frac{1-x}{x}} \quad (127)$$

is increasing in  $x \in [\alpha * \epsilon, 1/2]$ , the inequality in (126) holds. We will prove this by showing that the derivative of (127) in  $x$  is positive for the very noisy BSC of  $\epsilon \approx 1/2$ . The derivative of (127) is positive iff

$$\frac{-1}{(x-c)(1-(x-c))} \log \frac{1-x}{x} + \frac{1}{x(1-x)} \log \frac{1-(x-c)}{(x-c)} > 0. \quad (128)$$

From  $\alpha * \epsilon = \epsilon + \alpha(1-2\epsilon)$ , when  $\epsilon \approx 1/2$  it is implied that  $c = \alpha * \epsilon - \epsilon \approx 0$  and  $\alpha * \epsilon \approx 1/2$ . Therefore, in the regime of  $\alpha * \epsilon \leq x \leq 1/2$ ,  $c/x \approx 0$  and  $c/(1-x) \approx 0$ . We can approximate the terms in the left-hand side of (128) as

$$\begin{aligned} \log \frac{1-(x-c)}{(x-c)} &= \log \frac{(1-x) \left(1 + \frac{c}{1-x}\right)}{x \left(1 - \frac{c}{x}\right)} = \log \frac{1-x}{x} + \frac{c}{1-x} + \frac{c}{x} + O(c^2), \\ \frac{-1}{(x-c)(1-(x-c))} &= \frac{-1}{x(1-x)} \frac{1}{\left(1 - \frac{c}{x}\right) \left(1 + \frac{c}{1-x}\right)} = \frac{-1}{x(1-x)} \left(1 + \frac{c(1-2x)}{x(1-x)} + O(c^2)\right). \end{aligned} \quad (129)$$

By plugging these approximations, the left-hand side of (128) is approximated as

$$\frac{c}{x^2(1-x)^2} \left(1 - (1-2x) \log \frac{1-x}{x}\right) + O(c^2). \quad (130)$$

Since  $0 < (1-2x) \log \frac{1-x}{x} \ll 1$  for  $x = 1/2 - \delta$  for an arbitrarily small  $\delta > 0$ , it can be shown that (130) is positive. This implies that (128) is valid for the very noisy channel, and thus (126) is true. This concludes the proof for the statement 3)  $D_{\text{B}}(\gamma_{\text{GV}}(R_1) \|\alpha * \epsilon) > D_{\text{B}}(\gamma_{\text{GV}}(R_1 + C_2(\alpha)) \|\epsilon)$  for  $R_{\text{crit}}(\alpha * \epsilon) < R_1 < H_{\text{B}}(1/2) - H_{\text{B}}(\alpha * \epsilon)$ .

Lastly, we prove the statement 2)  $E_0(1/2, \alpha * \epsilon) - R_1 > D_{\text{B}}(\gamma_{\text{GV}}(R_1 + C_2(\alpha)) \|\epsilon)$  for  $0 \leq R_1 \leq R_{\text{crit}}(\alpha * \epsilon)$  when  $R_{\text{crit}}(\epsilon) < C_2(\alpha)$ . Statement 3) implies that at  $R_1 = R_{\text{crit}}(\alpha * \epsilon)$ ,  $E_0(1/2, \alpha * \epsilon) - R_1 > D_{\text{B}}(\gamma_{\text{GV}}(R_1 + C_2(\alpha)) \|\epsilon)$ , since  $E_0(1/2, \alpha * \epsilon) - R_1 = D_{\text{B}}(\gamma_{\text{GV}}(R_1) \|\alpha * \epsilon)$  at  $R_1 = R_{\text{crit}}(\alpha * \epsilon)$ . When  $R_{\text{crit}}(\epsilon) < C_2(\alpha)$ , the derivative of  $D_{\text{B}}(\gamma_{\text{GV}}(R_1 + C_2(\alpha)) \|\epsilon)$  in  $R_1 \in [0, R_{\text{crit}}(\alpha * \epsilon)]$  is

$$-1 < \frac{\partial}{\partial R_1} D_{\text{B}}(\gamma_{\text{GV}}(R_1 + C_2(\alpha)) \|\epsilon) \leq 0. \quad (131)$$

On the other hand, the derivative of  $(E_0(1/2, \alpha * \epsilon) - R_1)$  in  $R_1$  is  $\frac{\partial}{\partial R_1} (E_0(1/2, \alpha * \epsilon) - R_1) = -1$ . Since  $(E_0(1/2, \alpha * \epsilon) - R_1)$  decreases faster than  $D_{\text{B}}(\gamma_{\text{GV}}(R_1 + C_2(\alpha)) \|\epsilon)$  in  $R_1 \in [0, R_{\text{crit}}(\alpha * \epsilon)]$ , while  $(E_0(1/2, \alpha * \epsilon) - R_1)$  is still greater than  $D_{\text{B}}(\gamma_{\text{GV}}(R_1 + C_2(\alpha)) \|\epsilon)$  at  $R_1 = R_{\text{crit}}(\alpha * \epsilon)$ , it is implied that

$$E_0(1/2, \alpha * \epsilon) - R_1 > D_{\text{B}}(\gamma_{\text{GV}}(R_1 + C_2(\alpha)) \|\epsilon) \quad (132)$$

in  $R_1 \in [0, R_{\text{crit}}(\alpha * \epsilon)]$ .

We proved the three statements 1), 2) and 3), and these three statements imply the Lemma 6.



## APPENDIX F

## PROOF OF LEMMA 8: A LOWER BOUND ON THE ERROR EXPONENT OF MSBs WITH SUPERPOSITION CODING USING JOINT ML DECODER

In Lemma 8, we show that the average decoding error probability of  $M_1$  for the superposition code  $\{\mathbf{z}^{(m_1, m_2)}\}$ ,  $m_1 \in \{0, \dots, e^{NR_1} - 1\}$ ,  $m_2 \in \{0, \dots, e^{NR_2} - 1\}$ , with joint ML decoding rule achieves the partial decoding error probability of  $M_1$ ,  $\Pr(\hat{M}_{1, \text{JML}} \neq M_1) \leq e^{-N E_{\text{MSBs, JML}}^{\text{LB}}(R_1, R_2)}$  where

$$E_{\text{MSBs, JML}}^{\text{LB}}(R_1, R_2) = \begin{cases} E_0(1/2, \epsilon) - R_2 - R_1, & R_1 < \max\{0, R_{\text{crit}}(\epsilon) - R_2\}, \\ D_B(\gamma_{\text{GV}}(R_1 + R_2) \parallel \epsilon), & \max\{0, R_{\text{crit}}(\epsilon) - R_2\} < R_1 < H_B(1/2) - H_B(\epsilon) - R_2. \end{cases} \quad (133)$$

From the symmetry of the superposition code over messages  $m = (m_1, m_2)$ , the average decoding error probability of the partial message  $m_1$ , chosen uniformly at random from  $\{0, \dots, e^{NR_1} - 1\}$ , is equal to the decoding error probability of each partial message  $m_1 \in \{0, \dots, e^{NR_1} - 1\}$ . Therefore, without loss of generality, we suppose that  $\mathbf{z}^{(0,0)}$  is the correct codeword, which is transmitted over a BSC( $\epsilon$ ), and analyze the decoding error probability of  $m_1 = 0$ . Given the received word  $\mathbf{y} = \mathbf{z}^{(0,0)} \oplus \mathbf{n}$ , the joint maximum likelihood decoding rule finds a unique codeword  $\mathbf{z}^{(\hat{m}_1, \hat{m}_2)}$  that is closest to  $\mathbf{y}$ . When we denote the decoded message as  $(\hat{m}_1, \hat{m}_2)$ , the decoding error happens only when  $\hat{m}_1 \neq 0$ , regardless of whether  $\hat{m}_2 = 0$  or not.

The decoding error event  $\mathcal{E}_{\text{JML}}$  occurs if there exists a codeword  $\mathbf{z}^{(m_1, m_2)}$  with  $m_1 \neq 0$  whose distance from  $\mathbf{y}$  is less than or equal to the minimum of all distances between  $\mathbf{y}$  and  $\mathbf{z}^{(m_1, m_2)}$  for  $m_1 = 0$ ; i.e., when the minimum distance between  $\mathbf{y}$  and any incorrect codeword  $\mathbf{z}^{(m_1, m_2)}$  with  $m_1 \neq 0$  is  $N\delta$  and the minimum distance between  $\mathbf{y}$  and any codeword  $\mathbf{z}^{(m_1, m_2)}$  with  $m_1 = 0$  is  $N\tau$ , the decoding error happens with the event  $\mathcal{E}_{\text{JML}} = \{\delta \leq \tau\}$ .

Because only the distance of codewords from  $\mathbf{y}$  matters, we consider the “output-centered analysis” proposed in [15] where all codewords are translated by  $\mathbf{y}$ . Let  $\mathbf{w}^{(m_1, m_2)} = \mathbf{z}^{(m_1, m_2)} \oplus \mathbf{y} = \mathbf{z}^{(m_1, m_2)} \oplus \mathbf{z}^{(0,0)} \oplus \mathbf{n}$  denote the translated codewords. The translated correct codeword  $\mathbf{w}^{(0,0)}$  is the channel noise word  $\mathbf{n}$  and is independent of  $\mathbf{y}$ . The set of translated codewords for  $m_1 = 0$ ,  $\mathbf{w}^{(0, m_2)} = \mathbf{v}^{(m_2)} \oplus \mathbf{v}^{(0)} \oplus \mathbf{n}$  are independent of  $\mathbf{y}$  but dependent on  $\mathbf{n}$ . Moreover,  $\{\mathbf{w}^{(0, m_2)}\}$ ,  $m_2 \in \{0, \dots, e^{NR_2} - 1\}$  are dependent to each other. The rest of the translated codewords with  $m_1 \neq 0$ ,  $\{\mathbf{w}^{(m_1, m_2)}\}$ ,  $m_1 \in \{1, \dots, e^{NR_1} - 1\}$ ,  $m_2 \in \{0, \dots, e^{NR_2} - 1\}$  are independent of  $\mathbf{y}$ ,  $\mathbf{n}$ , and  $\{\mathbf{w}^{(0, m_2)}\}$ ,  $m_2 \in \{0, \dots, e^{NR_2} - 1\}$ . However, the codewords  $\{\mathbf{w}^{(m_1, m_2)}\}$ ,  $m_2 \in \{0, \dots, e^{NR_2} - 1\}$ , for a fixed  $m_1$  are dependent to each other. Lastly, all received words  $\mathbf{y}$  are equiprobable:  $p(\mathbf{y}) = 2^{-N}$ . The probability distribution of the system consisting of the translated codewords and a received word  $\mathbf{y}$  is thus

$$p(\{\mathbf{w}^{(0, m_2)}\}, \mathbf{y}, \{\mathbf{w}^{(m_1 \neq 0, m_2)}\}) = 2^{-N} p(\{\mathbf{w}^{(0, m_2)}\}) p(\{\mathbf{w}^{(m_1 \neq 0, m_2)}\}). \quad (134)$$

Therefore, we can think of the whole system as the one consisting of two independent subsystems, one comprising the translated codewords with  $m_1 = 0$  and the other the translated codewords with  $m_1 \neq 0$ . We analyze the decoding error probability of  $M_1$  with the JML decoding rule by using the independency between  $\{\mathbf{w}^{(0, m_2)}\}$  and  $\{\mathbf{w}^{(m_1 \neq 0, m_2)}\}$ .

The decoding error event  $\mathcal{E}_{\text{JML}}$  occurs if the minimum weight  $N\tau$  of  $\{\mathbf{w}^{(0,m_2)}\}$  is greater than or equal to the minimum weight  $N\delta$  of  $\{\mathbf{w}^{(m_1 \neq 0, m_2)}\}$ , i.e.,  $\mathcal{E}_{\text{JML}} = \{\delta \leq \tau\}$ . Define an error event  $\mathcal{E}_\gamma = \{\delta \leq \gamma \leq \tau\}$  for a fixed  $\gamma \in \Gamma = \{\gamma : 0 \leq \gamma \leq 1, N\gamma \in \mathcal{N}_0\}$  for the non-negative integer set  $\mathcal{N}_0$ . The decoding error probability is equal to  $\Pr(\mathcal{E}_{\text{JML}}) = \Pr(\delta \leq \tau) = \sum_{\gamma \in \Gamma} \Pr(\mathcal{E}_\gamma)$ . We first analyze  $\Pr(\mathcal{E}_\gamma)$  and then find the typical  $\gamma$  that dominates the error exponent for  $\Pr(\mathcal{E}_{\text{JML}})$ .

From the independency between  $\{\mathbf{w}^{(0,m_2)}\}$  and  $\{\mathbf{w}^{(m_1 \neq 0, m_2)}\}$ ,

$$\Pr(\mathcal{E}_\gamma) = \Pr(\delta \leq \gamma) \Pr(\tau \geq \gamma). \quad (135)$$

We first establish an upper bound on  $\Pr(\delta \leq \gamma)$ . Note that  $N\delta = \min_{(m_1 \neq 0, m_2)} w_H(\mathbf{w}^{(m_1, m_2)})$  where  $w_H(\cdot)$  is the Hamming weight of the sequence. We need to analyze the distribution of  $w_H(\mathbf{w}^{(m_1 \neq 0, m_2)})$ . Note that every symbol of each  $\mathbf{w}^{(m_1 \neq 0, m_2)}$  is equiprobable. By the Chernoff exponent lemma, for  $\gamma < 1/2$ , the probability that the Hamming weight  $w_H(\mathbf{w}^{(m_1 \neq 0, m_2)})$  of a given incorrect codeword  $\mathbf{w}^{(m_1 \neq 0, m_2)}$  is less than or equal to  $N\gamma$  is

$$\Pr(w_H(\mathbf{w}^{(m_1 \neq 0, m_2)}) \leq N\gamma) \doteq e^{-ND_B(\gamma\|1/2)}. \quad (136)$$

The event  $\delta \leq \gamma$  occurs when there exists a  $(m_1 \neq 0, m_2)$  such that  $w_H(\mathbf{w}^{(m_1 \neq 0, m_2)}) \leq N\gamma$ . Since there are  $(e^{NR} - e^{NR_2})$  codewords with  $m_1 \neq 0$ , by the union bound

$$\Pr\left(\min_{(m_1 \neq 0, m_2)} w_H(\mathbf{w}^{(m_1 \neq 0, m_2)}) \leq N\gamma\right) \leq \begin{cases} e^{-N(D_B(\gamma\|1/2) - R)}, & \gamma \leq \gamma_{\text{GV}}(R), \\ 1, & \gamma > \gamma_{\text{GV}}(R). \end{cases} \quad (137)$$

We next analyze  $\Pr(\tau \geq \gamma)$  where  $\tau = \min_{(m_1=0, m_2)} w_H(\mathbf{w}^{(m_1, m_2)})$ . The translated correct codeword  $\mathbf{w}^{(0,0)}$  is equal to  $\mathbf{n}$  and is distributed by

$$p(\mathbf{n}) = \epsilon^{w_H(\mathbf{n})} (1 - \epsilon)^{N - w_H(\mathbf{n})}. \quad (138)$$

Therefore, for  $\gamma > \epsilon$ , by the Chernoff exponent lemma we have

$$\Pr(w_H(\mathbf{n}) \geq N\gamma) \doteq e^{-ND_B(\gamma\|\epsilon)}. \quad (139)$$

The event  $\tau \geq \gamma$  occurs when the Hamming weight  $w_H(\mathbf{w}^{(m_1=0, m_2)})$  of every codeword  $\{\mathbf{w}^{(m_1=0, m_2)}, 0 \leq m_2 \leq e^{NR_2} - 1\}$  is greater than or equal to  $N\gamma$ . Therefore, for  $\epsilon < \gamma < 1/2$ ,

$$\Pr\left(\min_{(m_1=0, m_2)} w_H(\mathbf{w}^{(m_1=0, m_2)}) \geq N\gamma\right) \leq \Pr(w_H(\mathbf{n}) \geq N\gamma) \doteq e^{-ND_B(\gamma\|\epsilon)}. \quad (140)$$

This bound may not be exponentially tight for  $0 \leq \alpha < 1/2$ . However, when  $\alpha = 1/2$ , since  $\{\mathbf{w}^{(0, m_2 \neq 0)}, \mathbf{n}\}$  are independent to each other and every symbol of every  $\mathbf{w}^{(0, m_2 \neq 0)}$  is independent and equiprobable, for  $\epsilon < \gamma < 1/2$

$$\Pr(w_H(\mathbf{w}^{(0, m_2 \neq 0)}) \geq N\gamma) \doteq 1. \quad (141)$$

Therefore, for the case of  $\alpha = 1/2$ , the upper bound in (140) becomes exponentially tight.

From (136) and (140),

$$\Pr(\mathcal{E}_\gamma) \leq \begin{cases} e^{-N(D_B(\gamma\|\epsilon) + D_B(\gamma\|1/2) - R)}, & \epsilon < \gamma \leq \gamma_{\text{GV}}(R), \\ e^{-ND_B(\gamma\|\epsilon)}, & \gamma > \gamma_{\text{GV}}(R). \end{cases} \quad (142)$$

By using this result, we calculate the achievable exponent of  $\Pr(\mathcal{E}_{\text{JML}}) = \sum_{\gamma \in \Gamma} \Pr(\mathcal{E}_{\gamma})$  by finding  $\gamma$  that dominates the exponentially decreasing rate of  $\Pr(\mathcal{E}_{\text{JML}})$  in  $N$ . The resulting  $\Pr(\mathcal{E}_{\text{JML}})$  is

$$\Pr(\mathcal{E}_{\text{JML}}) \stackrel{\cdot}{\leq} \begin{cases} e^{-N(E_0(1/2, \epsilon) - R)}, & R < R_{\text{crit}}(\epsilon), \\ e^{-ND_{\text{B}}(\gamma_{\text{GV}}(R) \parallel \epsilon)}, & R_{\text{crit}}(\epsilon) \leq R < C = H_{\text{B}}(1/2) - H_{\text{B}}(\epsilon), \end{cases} \quad (143)$$

where  $E_0(a, b) = -\log(1 - 2a(1 - a)(\sqrt{b} - \sqrt{1 - b})^2)$  and thus  $E_0(1/2, \epsilon) = \log 2 - \log(1 + 2\sqrt{\epsilon(1 - \epsilon)})$ , and  $R_{\text{crit}}(\epsilon) = D_{\text{B}}(\gamma_{\text{crit}}(\epsilon) \parallel 1/2)$  with  $\gamma_{\text{crit}}(\epsilon) = \frac{\sqrt{\epsilon}}{\sqrt{\epsilon} + \sqrt{1 - \epsilon}}$ .

Therefore, the achievable error exponent  $E_{\text{MSBs, JML}}^{\text{LB}}(R_1, R_2)$  of the superposition code of total rate  $R = R_1 + R_2$  with the joint ML decoder is

$$E_{\text{MSBs, JML}}^{\text{LB}}(R_1, R_2) = \begin{cases} E_0(1/2, \epsilon) - R_2 - R_1, & R_1 < \max\{0, R_{\text{crit}}(\epsilon) - R_2\}, \\ D_{\text{B}}(\gamma_{\text{GV}}(R_1 + R_2) \parallel \epsilon), & \max\{0, R_{\text{crit}}(\epsilon) - R_2\} \leq R_1 < H_{\text{B}}(1/2) - H_{\text{B}}(\epsilon) - R_2. \end{cases} \quad (144)$$

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